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Équations aux dérivées partielles/*Partial Differential Equations*

On a two-dimensional stationary free boundary problem arising in the confinement of a plasma in a Stellarator

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Abstract – We obtain the existence of solutions of a two-dimensional free boundary problem modelling the magnetic confinement of a plasma in a Stellarator configuration. The nonlinear elliptic equation was obtained from the 3-D MHD system by Hender and Carreras using averaging arguments and a suitable system of coordinates. The final formulation as a free boundary problem involves the notion of relative rearrangement and has important differences with respect to the model for Tokamaks configurations. We use an iterative algorithm and some new properties on relative rearrangement in order to prove the existence of a solution giving rise to a free boundary.

Sur un problème stationnaire bidimensionnel à frontière libre issu du confinement d'un plasma dans un Stellarator

Résumé – *Nous montrons l'existence de solutions d'un problème stationnaire à frontière libre modélisant le confinement magnétique d'un plasma dans un Stellarator. L'équation elliptique non linéaire a été obtenue à partir du système de la MHD à 3 dimensions par Hender et Carreras en utilisant des arguments de moyennisation et un système de coordonnées convenable. La formulation finale comme problème de frontière libre fait intervenir la notion de réarrangement relatif. Cette formulation est très différente de celle obtenue pour des Tokamaks. Par un algorithme itératif et l'usage de nouvelles propriétés sur le réarrangement relatif, nous montrons l'existence d'une solution à frontière libre.*

Version française abrégée – Étant donnés Ω un ouvert borné régulier et connexe de $\mathbb{R}^2$, a, b, λ et F_v satisfaisant (2), pour chaque $\gamma \in \mathbb{R}$ fixé on considère le problème (P)

$$(\mathrm{P})\quad \left| \begin{aligned} & -\Delta u = a\left[F_v^2 - \lambda\int_{|u>0|}^{|u>u_+(x)|} [(u_*)_+^2]'(\sigma)\, b_{*u}(\sigma)\, d\sigma\right]_+^{1/2} + \lambda u_+ [b - b_{*u}(|u>u(x)|)] \\ & u = \gamma \quad \text{sur } \partial\Omega, \end{aligned}\right.$$

où le terme b_{*u} représente le réarrangement relatif de b par rapport à u et u_* le réarrangement décroissant de u. Ce type de problème apparaît dans le confinement magnétique d'un plasma en équilibre dans une configuration Stellarator. Le passage de la formulation 3-D du système (1) de la MHD à la formulation bidimensionnelle a été fait dans [8] en utilisant des arguments de moyennisation. La formulation comme problème à frontière libre a été introduite dans [4] (*voir* aussi [6]). Bien que la formulation physique amène à un opérateur $\mathscr{L}$ plus compliqué que le Laplacien, nos techniques peuvent être aussi utilisées dans ce cas en introduisant des espaces à poids [7]. Notre résultat principal est le suivant :

THÉORÈME. – *Soit λ_1 la première valeur propre du laplacien sur Ω avec la condition de Dirichlet et soit ϕ_1 la fonction propre associée telle que $\int \Delta\phi_1\, dx = -1$. Soit $\gamma \in \mathbb{R}$ fixé. Alors il existe $\delta > 0$, indépendant de λ, tel que si*

$$(\mathrm{H})\qquad \lambda \| b \|_{L^\infty(\Omega)} < \inf (\lambda_1/4,\ \delta F_v,\ \inf |a|^2),$$

le problème (P) *a une solution $u \in W^{2,p}(\Omega)$ [pour tout $p \in [1, \infty)$] tel que*

$$|\{x \in \Omega : \nabla u(x) = 0\}| = 0.$$

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De plus, $u_+ \not\equiv 0$, *si* $-\gamma < F_v \int a(x)\,\phi_1(x)\,dx$.

Idée de la démonstration. — Étant fixé $\gamma \in \mathbb{R}$, $w^0 \in C^0(\bar{\Omega})$ et $j \in \mathbb{N}$ on considère la suite des problèmes (P^j) (*voir* la version anglaise). Grâce à une propriété du réarrangement relatif (*voir* Lemma 2) on a G^j, $J^j \in L^\infty(\Omega)$. Par induction on trouve l'existence de $w^j \in W^{2,p}(\Omega)$ [pour tout $p = [1, \infty)$], solution de (P^j) satisfaisant (11). Les estimations *a priori* s'obtiennent grâce à (H) en multipliant par w^j (*voir* Lemma 3) et en dérivant une relation de récurrence. Un argument dû à Stampacchia permet de montrer que (H) implique que $|\{x \in \Omega : \nabla w^j(x) = 0\}| = |\{x \in \Omega : \nabla w(x) = 0\}| = 0$ (w est une limite de w^j). Le passage à la limite est fait en utilisant la continuité du réarrangement pour des fonctions *co-aire régulières* [au sens de Almgren-Lieb [1]; *voir* aussi Rakotoson [12]) ainsi que certaines propriétés du réarrangement relatif. La conclusion $u_+ \not\equiv 0$ se réduit par un argument similaire à celui fait dans Diaz [5]).

Les détails, généralisations et autres propriétés qualitatives seront donnés dans Diaz-Rakotoson [7].

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1. MODELLING. STATEMENT OF THE PROBLEM. — The static equilibrium magnetohydrodynamic system (MDH) on the pressure p, the magnetic field $\mathbf{B}$ and the current $\mathbf{J}$ are

$$\mathbf{J} \times \mathbf{B} = \nabla p, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \qquad \nabla . \mathbf{B} = 0. \tag{1}$$

The crucial fact in magnetic fusion is the analysis of different magnetic geometries. In the Tokamak case the toroidal configuration is axisymetric, the equilibium is independent of the toroidal angle and (1) leads to a two-dimensional problem. The mathematical treatement of the associated free-boundary formulation attracted the attention of numerous authors: Temam [15], Berestycki-Brezis [3] and many others (*see* some references in the monograph Blum [2]).

More complicated three-dimensional configurations, like the Stellarators, do not need any toroidal current on the plasma. Hender-Carreras [8] reduce the problem to a two-dimensional equation by an averaging method applied after introducing the Boozer vacuum flux coordinates (ρ, θ, ϕ) "playing the role" of the radial coordinate and the poloidal and toroidal angle respectively. The resulting Grad-Shafranov type equation for the averaged poloidal flux function u holds only on the plasma region (which is *a priori* unknown). To overcome this difficulty, in Diaz [4] (*see* also [6]) was introduced the following free boundary value problem: let Ω be an open bounded regular connected set of $\mathbb{R}^2$ and let

$$\lambda > 0, \qquad F_v > 0, \qquad a, b \in L^\infty(\Omega), \quad b > 0, \quad a.e. \text{ on } \Omega. \tag{2}$$

Given $\gamma \in \mathbb{R}$, find $u : \Omega \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}_+$ such that $F(s) = F_v$ for any $s \leqq 0$ and the following conditions hold

$$-\mathscr{L} u = a(\rho, \theta)\,F(u) + F(u)\,F'(u) + \lambda\, b(\rho, \theta)\, u_+ \quad \text{in } \Omega, \tag{3}$$

$$u = \gamma \quad \text{on } \partial\Omega, \tag{4}$$

$$\int_{\{u \geqq t\}} [F(u)\,F'(u) + \lambda\, b u_+]\,\rho\, d\rho\, d\theta = 0 \quad \text{for any } t \in [\inf u, \sup u]. \tag{5}$$

Here $\mathscr{L}$ represents the second order partial differential operator obtained in Hender-Carreras [2]

$$\mathscr{L}u(\rho, \theta)=\frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(a_{\rho\rho}\frac{\partial u}{\partial \rho}\right)+\frac{\partial}{\partial \rho}\left(a_{\rho\theta}\frac{\partial}{\partial \theta}\right)+\frac{\partial}{\partial \theta}\left(a_{\theta\rho}\frac{\partial u}{\partial \rho}\right)+\frac{\partial}{\partial \theta}\left(a_{\theta\theta}\frac{\partial u}{\partial \theta}\right)\right]$$

where the coefficients are determined from the geometry of the vacuum magnetic surfaces. The uniform ellipticity of $\mathscr{L}$ was shown in Diaz [4]. In (3) we are assuming the usual constitutive law $p=p(u)=\lambda(u_+)^2/2$, $u_+=\max(u, 0)$. The boundary condition (4) comes from the fact that $\langle \mathbf{B}\rangle.\mathbf{n}=0$ where $\mathbf{n}$ is the outer unit normal vector. Finally, conditions (5) express the Stellarator condition that the total current within each flux surface must be zero. We point out that the term $\mathrm{F}(u)$ represents the averaged toroidal covariant coordinate of $\mathbf{B}$ and so it is an unknown of the problem. We remark that some numerical algorithms start by approching u assumed F given (the mathematical treatment of this intermediate problem was done in Diaz [4], [5], [6]).

In order to eliminate the unknown F, problem (3), (4), (5) was reformulated in Diaz [5] by using the notion of relative rearrangement introduced in Mossino-Temam [9]. Following the arguments of [5], [6], it is possible to show that any solution u of (3), (4), (5), verifying $u\in \mathrm{W}^{2,p}(\Omega)$ [for any $p\in[1, \infty)$] and $|\{x\in\Omega:\nabla u(x)=0\}|=0$, must satisfy the following non local equation

$$(6)\quad -\mathscr{L}u=a\left[\mathrm{F}_v^2-\lambda\int_{|u>0|}^{|u>u_+(\rho,\theta)|}[(u_*)_+^2]'(\sigma)\,b_{*_u}(\sigma)\,d\sigma\right]_+^{1/2} +\lambda u_+[b-b_{*_u}(|u>u(\rho,\theta)|)]$$

where u_* represents the decreasing rearrangement of u and b_{*_u} is the relative rearrangement of b with respect to u. We recall that given $h:\Omega\to\mathbb{R}$ we define *the distribution function of* h, $m(t)=|h>t|=\operatorname{meas}\{(\rho, \theta)\in\Omega:h(\rho, \theta)>t\}$ and then *the decreasing rearrangement of* h, h_*: $(0, |\Omega|)\to\mathbb{R}$ by $h_*(s)=\inf\{t\in\mathbb{R}:m(t)\leqq s\}$. Finally, if $b\in \mathrm{L}^1(\Omega)$ we define *the relative rearrangement of* b *with respect to* h by $b_{*_h}(s)=(dw/ds)(s)$ $s\in]0, |\Omega|[$, where

$$w(s)=\int_{\{h>h_*(s)\}}b(\rho, \theta)\,d\rho\,d\theta+\int_0^{s-|h>h_*(s)|}(b_{|\mathrm{P}(s)})_*(\sigma)\,d\sigma$$

with $\mathrm{P}(s):=\{(\rho, \theta)\in\Omega:h(\rho, \theta)=h_*(s)\}$(*see* more details in [9], [10], [11]). It is not difficult to show that, conversely, such a solution u of (6), (4) also satisfies (3) (5) for a suitable function F.

The main goal of this Note is to show that, under suitable assumptions, problem (4) (6) has a solution u satisfying other properties of interest in the physical formulation; mainly that if $\gamma<0$ then $u_+\not\equiv 0$ (*i. e.* u gives rise to a free boundary). In order to simplify the exposition, in the following section we shall limit ourselves to the study of the case $\mathscr{L}=\Delta$. We point out that the operator $\mathscr{L}$ can be treated in a similar way after introducing suitable weighted-Sobolev spaces. The details will be given in Diaz-Rakotoson [7].

2. On the existence of solutions. — Given a, b, λ and F_v satisfying (2) and given $\gamma\in\mathbb{R}$ we consider the problem of finding $u:\Omega\to\mathbb{R}$ satisfying (P) (*see* its formulation in the french version). The main result of this Note is the following

Theorem. — *Let* λ_1 *be the first eigenvalue and let* ϕ_1 *the associated eigenfunction of the Laplacian on* Ω *such that* $\phi_1=0$ *on* $\partial\Omega$ *and* $\int\Delta\phi_1=-1$. *We fix* $\gamma\in\mathbb{R}$. *There exists*

$\delta>0$, *independent of* λ, *such that if*

(H) $$\lambda \| b \|_{L^\infty(\Omega)} < \inf (\lambda_1/4, \delta F_v, \inf |a|^2),$$

problem (P) *has a solution* $u \in W^{2,p}(\Omega)$ (*for any* $p \in [1, \infty)$) *such that*

$$|\{x \in \Omega : \nabla u(x) = 0\}| = 0.$$

Moreover $u_+ \neq 0$ *assumed that*

(7) $$-\gamma < F_v \int_\Omega a(x)\, \phi_1(x)\, dx.$$

Sketch of the Proof. — We shall proceed in several steps.

1*st step.* — *Formulation of the approximate problems.* — Given $\gamma \in \mathbb{R}$, $w^0 \in C^\infty(\bar{\Omega})$ and $j \in \mathbb{N}$ we consider the problem

(P^j) $$-\Delta w^{j+1} = a(x)\, G^j(x) + J^j(x) \quad \text{in } \Omega, \qquad w^{j+1} = 0 \quad \text{on } \partial\Omega,$$

with

$$G^j(x) := \left[F_v^2 - \lambda \int_{|w^j+\gamma>0|}^{|w^j+\gamma>(w^j(x)+\gamma)_+|} ([(w^j+\gamma)_*]_+^2)'(\sigma)\, b_{*_{w^j}}(\sigma)\, d\sigma \right]_+^{1/2},$$

$$J^j(x) = \lambda (w^j(x)+\gamma)_+ [b(x) - b^{w^j}(x)], \quad b^{w^j}(x) := \frac{(d^+/ds)(w^j)_*(|w^j > w^j(x)|)}{(d^+/ds)(w^j)_*^b(|w^j > w^j(x)|_b) - (1/(j+1))},$$

where $|w^j > w^j(x)|_b = \int_{|w^j > w^j(x)|} b(y)\, dy$ and $(w^j)_*^b$ is the decreasing rearrangement of w_j replacing the Lebesque measure dx by the weighted measure $b(x)\, dx$ (*i.e.* $(w^j)_*^b(s) = \inf \{t \in \mathbb{R} \mid |w^j > t|_b \leqq s\}$). The motivation of introducing the term b^{w^j} comes from the following result

LEMMA 1. — *Let* $z \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ *such that* $|\{x \in \Omega : \nabla z(x) = 0\}| = 0$. *Then*

(8) $$b_{*_z}(|z > z(x)|) = [z_*]'(|z > z(x)|)/[z_*^b]'(|z > z(x)|_b) \quad a.e.\ x \in \Omega.$$

Ideal of the proof. — It follows from the application of the general Federer co-area formula as in Rakotoson [10]. ■

LEMMA 2. — *Let* $z \in W^{2,s}(\Omega)$ *for some* $s > 1$. *Then*

(9) $$\left| \frac{d^+ z_*}{ds}(|z > z(x)|) \right| \leqq \| b \|_{L^\infty(\Omega)} \left| \frac{d^+ z_*^b}{ds}(|z > z(x)|_b) \right| \qquad \textit{for} \quad a.e.\ x \in \Omega.$$

Idea of the proof. — We approximate z by $\{p_n\}$ satisfying the assumptions of Lemma 1 (e.g. p_n an analytic function) and such that $p_n \to z$ in $W^{1,q}(\Omega)$ for some $q > 2$. Then Lemma 1 and the property $\| b_{*_z} \|_{L^\infty(\Omega)} \leqq \| b \|_{L^\infty(\Omega)}$ (*see* Mossino-Temam [9]) gives (9) for p_n. Finally as z is a *co-area regular* function by Almgren-Lieb [1] we can pass to the limit as $n \to \infty$ and (9) holds. ■

2*nd step.* — A priori *estimates.* — Let $j = 0$. Applying Lemma 2 to $z = w^0$ we have that G^0, $J^0 \in L^\infty(\Omega)$. Then there exists a unique function $w^1 \in W^{2,p}(\Omega)$ [for any $p \in [1, \infty)$] solution of (P^0). Applying again Lemma 2 we get that $J^1 \in L^\infty(\Omega)$ [it is obvious that $G^j \in L^\infty(\Omega)$ for any j] and so by induction we get the existence of a unique solution $w^j \in W^{2,p}(\Omega)$ of (P^j) satisfying also

(10) $$\| b^{w_j} \|_{L^\infty(\Omega)} \leqq \| b \|_{L^\infty(\Omega)} \quad \text{for any } j \in \mathbb{N}.$$

LEMMA 3. — *Assume* (H). *Then for all p in* $[1, \infty)$ *there exists* M *such that*

(11) $$\limsup_j \| w^i + \gamma \|_{W^{2,p}(\Omega)} \leqq M.$$

Idea of the proof. — We have that $\| G^j \|_{L^\infty(\Omega)} \leqq F_v$. By Lemma 2

(12) $$| J^j(x) | \leqq 2\lambda \| b \|_{L^\infty(\Omega)} (w^j(x) + \gamma)_+ \quad \text{for } a.e.\ x \in \Omega.$$

Multiplying the equation of (P^j) by w^{j+1} and using the Hölder inequality we obtain that if $d^j = \| w^j + \gamma \|_{L^2(\Omega)}$ then there exist $\varepsilon \in (0, 1/2)$ and $\tilde{M} > 0$ such that $d^{j+1} \leqq \varepsilon d^j + \tilde{M}$ which implies $\limsup_j d^j \leqq 2\tilde{M}$. Using (12) and (P^j) we obtain a uniform estimate in $W^{2,2}(\Omega)$ and by a standard boostrap argument we get (11) ■

3*rd step.* — *Passing to the limit.* — Let w, G and J be such that $w^j \to w$ weakly in $W^{2,p}(\Omega)$ (for all $p \in (1, \infty)$), $G^j \to G$ and $J^j \to J$ weak * in $L^\infty(\Omega)$. Obviously, $w = 0$ on $\partial\Omega$ and $-\Delta w = a G + J$. Moreover, since we have

$$\int_{| w^j+\gamma > 0 |}^{| w^j+\gamma > (w^j(x)+\gamma)_+ |} ([(w^j+\gamma)_*]_+^2)'(\sigma)\, b_{*_{w^j}}(\sigma)\, d\sigma \leqq \| b \|_{L^\infty(\Omega)} (w^j(x)+\gamma)_+^2 \qquad \text{for } \quad a.e.\ x \in \Omega.$$

then we can choose $\nu \in]0, 1[$ such that

(13) $$(1-\nu)^{1/2} F_v \leqq G^j(x) \quad a.e.\ x \in \Omega,$$

(14) $$(1-\nu)^{1/2} F_v \leqq G(x) \leqq F_v \qquad \text{and} \qquad \| J \|_{L^\infty(\Omega)} \leqq K \quad \text{for some K.}$$

Necessarily $| \{ x \in \Omega : \nabla w^j(x) = 0 \} | = | \{ x \in \Omega : \nabla w(x) = 0 \} | = 0$ since otherwise using an argument due to Stampacchia we would get a contradiction to (H). Now we shall verify that $u := w + \gamma$ satisfies (P). Since w is co-area regular, by a result due to Almgren-Lieb [1] and a slight generalization of [1], we get the pointwise convergence of $(w_*^j)'$ (resp. $[(w^j)_*^b]'$) to $(w_*)'$ (resp. $[w_*^b]'$) (*see* Rakotoson [12] and also [7]). Furthermore, from the boundedness of $(w_*^j)'$ [resp. $((w^j)_*^b)'$] in some $L^q(0, |\Omega|)$ [resp. in $L^q(0, |\Omega|_b)$] with $q > 1$ (*see* Rakotoson-Simon [13] and also [11]) we get the strong convergence. On the other hand we deduce from the result of [11] (*see* also [12]) that $b_{*_{w^j}}$ converges to b_{*_w} in $L^\infty(0, |\Omega|)$ weak $*$. Let χ_{I^j} be the characteristic function the set $I^j = (| w^j + \gamma > (w^j+\gamma)_+(x) |, | w^j > 0 |)$. Then $\chi_{I^j} \to \chi_I$ $a.e.$ where $I = (| w - \gamma > (w-\gamma)_+(x) |, | w > 0 |)$. It easy to check that $G^j(x)$ converges to

$$G(x) = \left[F_v^2 - \lambda \int_{| u > 0 |}^{| u > u_+(x) |} [(u_*)_+^2]'(\sigma)\, b_{*_u}(\sigma)\, d\sigma \right]_+^{1/2}.$$

Moreover by equimeasurability and the above convergences, we get

$$\| (w_*^j)'(| w^j > w^j(x) |) \|_{L^q(\Omega)} = \| (w_*^j)' \|_{L^q(0, |\Omega|)} \to \| (w_*)' \|_{L^q(0, |\Omega|)} = \| (w_*)'(| w > w(x) |) \|_{L^q(\Omega)}.$$

Using the averaging operators of Mossino-Temam [9] we deduce the weak convergence of $[(w^j)_*]'(| w^j > w^j(x) |)$ to $[w_*]'(| w > w(x) |)$ in $L^q(0, |\Omega|)$. By a result of Functional Analysis, the above convergence is strong. Similar arguments lead to the strong convergence of $[(w^j)_*^b]'(| w^j > w^j(x) |_b)$ to $[(w)_*^b]'(| w > w(x) |_b)$ in $L^q(0, |\Omega|_b)$. The convergence

$$b^{w^j}(x) \to b_{*_w}(| w > w(x) |) \quad \text{in } L^s(\Omega) \quad \text{for any } s \in [1, \infty)$$

comes from the Legesgue Theorem since we have the pointwise convergence (use the above convergences) and (10). This implies the convergence of $J^j(x)$ to $J(x) = \lambda u_+ [b - b_{*_u}(| u > u(x) |)]$ strongly in $L^s(\Omega)$ for any finite $s \geqq 1$. So, u satisfies (P).

Finally, multiplying the equation of (P) by ϕ_1 normalized such that $\int_\Omega \Delta\phi_1 = -1$, integrating by parts and arguing by contradiction we obtain that (H) implies that $u_+ \not\equiv 0$. ■

Remark. — A detailed version containing also additional results will be given in Diaz-Rakotoson [7]. So, for instance, under some more general assumption it is possible to show the existence of a solution satisfying (P) in a suitable weak sense. Other results on the dependence of the size of the plasma region $\{u_+ \not\equiv 0\}$ with respect to γ will be given.

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REFERENCES

[1] F. J. ALMGREN and E. LIEB, Symmetric decreasing rearrangement is sometimes continuous, *J. Amer. Math. Soc.*, 2, 1989, pp. 683-772.

[2] J. BLUM, *Numerical simulation and optimal control in Plasma Physics*, Wiley, Gauthier-Villars, Paris, 1989.

[3] H. BERESTYCKI and H. BREZIS, On a free boundary problem arising in plasma physics, *Nonlinear Anal.*, 4, 1980, pp. 415-436.

[4] J. I. DIAZ, *Modelos bidimensionales de equilibrio magnetohidrodinàmico para Stellarators. Informe #1. Formulación.* CIEMAT Repports. Madrid, December 1991, 38 p.

[5] J. I. DIAZ, *Modelos bidimensionales de equilibrio magnetohidrodinàmico para Stellarators, Informe #2, Existencia de soluciones*, CIEMAT Repports, Madrid, July 1992, 30 p.

[6] J. I. DIAZ, On an intermediate free boundary problem in the study of the magnetohydrodynamic equilibrium of a plasma in a Stellarator configuration in *Mathematical models in Plasma Physics*, J. I. DIAZ and A. GALINDO Ed., Real Academia de Ciencias, Madrid, 1993 (to appear).

[7] J. I. DIAZ and J.-M. RAKOTOSON, article in preparation.

[8] T. C. HENDER and B. A. CARRERAS, Equilibrium calculations for helical axis Stellarators, *Phys. Fluids*, 27, 1984, pp. 2101-2120.

[9] J. MOSSINO and R. TEMAM, Directional derivative of the increasing rearrangement mapping and application to a queer differential equation in Plasma Physics, *Duke Math. J.*, 48, 1983, pp. 475-495.

[10] J.-M. RAKOTOSON, Some properties of the relative rearrangement, *J. Math. Anal. Appl.*, 135, 1988, pp. 488-500.

[11] J.-M. RAKOTOSON, A differentiability result for the relative rearrangement, *Differential and Integral Equations*, 2, 1989, pp. 363-377.

[12] J.-M. RAKOTOSON, Relative rearrangement for highly nonlinear equations, *Nonlinear Analysis* (to appear).

[13] J.-M. RAKOTOSON and B. SIMON, Relative rearrangement on a measure space. Application to the regularity of weighted monotone rearrangement. Part. I. *Appl. Math. Lett.*, 6, 1983, pp. 75-78; Part. II. *Appl. Math. Lett.*, 6, 1993, pp. 79-82.

[14] J.-M. RAKOTOSON and R. TEMAM, A co-area formula with applications to monotone rearrangement and to regularity, *Arch. Rational Mech. Anal.*, 109, 1991, pp. 213-238.

[15] R. TEMAM, A nonlinear eigenvalue problem: equilibrium shape of a confined plasma, *Arch. Rational Mech. Anal.*, 60, 1975, pp. 51-73.

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