Reprinted from JOURNAL OF DIFFERENTIAL EQUATIONS
All Rights Reserved by Academic Press, New York and London

Vol. 106, No. 2, December 1993
Printed in Belgium

Existence and Regularity of Renormalized Solutions for Some Elliptic Problems Involving Derivatives of Nonlinear Terms

L. BOCCARDO

Dipartimento di Matematica, Università di Roma I, Piazzale A. Moro 2, 00185 Roma, Italy

D. GIACHETTI

Dipartimento di Matematica Applicata, Facoltà di Ingegneria, Via Scarpa 10, 00161 Roma, Italy

J. I. DIAZ

Departamento de Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain

AND

F. MURAT

Laboratoire d'Analyse Numérique, Université Paris VI, Tour 55-65, 5ème étage, 4 Place Jussieu, 75252 Paris Cedex 05, France

Received June 12, 1991

Contents.

0022-0396/93 $5.00

1. Results and Comments

1.1. *Definition and Existence of Renormalized Solutions*

Let Ω be a bounded open subset of $\mathbb{R}^N$ and p and p' belong to $]1, +\infty[$ with $1/p + 1/p' = 1$. In the following A will be a nonlinear operator of the Leray–Lions type from $W_0^{1,p}(\Omega)$ into its dual defined by

$$Au = -\operatorname{div}(a(x, u, Du)),$$

where $a(x, t, \xi)$ is a Caratheodory vector valued function from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}^N$ such that

$$\begin{cases} |a(x, t, \xi)| \leqslant c_1 |t|^{p-1} + c_2 |\xi|^{p-1} + d(x) \\ c_1, c_2 > 0, \qquad d(x) \in L^{p'}(\Omega), \\ \text{a.e. } x \text{ in } \Omega, \qquad \forall (t, \xi) \in \mathbb{R}^{N+1}, \end{cases} \tag{1.1}$$

$$\begin{cases} [a(x, t, \xi) - a(x, t, \bar{\xi})][\xi - \bar{\xi}] > 0 \\ \text{a.e. } x \text{ in } \Omega, \qquad \forall (t, \xi), (t, \bar{\xi}), \xi \neq \bar{\xi}, \end{cases} \tag{1.2}$$

$$\begin{cases} a(x, t, \xi)\xi \geqslant \alpha |\xi|^p, \qquad \alpha > 0 \\ \text{a.e. } x \text{ in } \Omega, \qquad \forall (t, \xi) \in \mathbb{R}^{N+1}. \end{cases} \tag{1.3}$$

In this paper we deal with the problem of existence and L^s-regularity of solutions for the following nonlinear elliptic problem

$$-\operatorname{div}(a(x, u, Du)) - \operatorname{div}(\Phi(u)) + g(x, u) = f \qquad \text{in } \Omega \tag{1.4}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1.5}$$

The right hand side of (1.4) and $\Phi = (\Phi_1, ..., \Phi_N)$ are assumed to satisfy

$$f \in W^{-1,p'}(\Omega) \tag{1.6}$$

$$\Phi \in (C^0(\mathbb{R}))^N. \tag{1.7}$$

Moreover the function $g(x, t)$ is a Caratheodory function satisfying

$$g(x, t)t \geqslant 0 \qquad \text{a.e. } x \text{ in } \Omega, \qquad \forall t \in \mathbb{R} \tag{1.8}$$

$$\operatorname*{Sup}_{|t| \leqslant n} |g(x, t)| = h_n(x) \in L^1(\Omega). \tag{1.9}$$

No growth hypothesis is assumed on the function Φ. This implies that, for a solution u in $W_0^{1,p}(\Omega)$ (which is the natural space in view of assumptions (1.1), (1.2), (1.3), and (1.6)), the term $\operatorname{div}(\Phi(u))$ may be meaningless, even as a distribution. The first objective of the present paper is thus to give a meaning to a possible solution of (1.4), (1.5). This will be done multiplying (1.4) by $h(u)$ where $h \in C_c^1(\mathbb{R})$ (the class of $C^1(\mathbb{R})$ functions with compact support). This gives rise to a weaker problem, which can actually be proved to have a solution.

The following Theorem will be proved in Section 2.

THEOREM 1. *Assume that* (1.1), (1.2), (1.3), (1.6), (1.7), (1.8), *and* (1.9) *hold. Then there exists a solution of*

$$u \in W_0^{1,p}(\Omega), \qquad g(x, u) \in L^1(\Omega), \qquad ug(x, u) \in L^1(\Omega) \tag{1.10}$$

$$\begin{cases} [-\operatorname{div}(a(x, u, Du))]\, h(u) - \operatorname{div}(\Phi(u)\, h(u)) + \Phi(u)\, h'(u)\, Du + g(x, u)\, h(u) \\ \quad = fh(u) \quad \text{in} \quad \mathscr{D}'(\Omega), \qquad \forall h \in C_c^1(\mathbb{R}). \end{cases} \tag{1.11}$$

A solution of (1.10)–(1.11) will be called the "renormalized solution" of the original problem (1.4)–(1.5).

Remark 1. Let us note that in (1.11) every term is meaningful in the distributional sense (in contrast with (1.4)). Indeed consider any φ in $\mathscr{D}(\Omega)$; then $h(u)\varphi$ belongs to $W_0^{1,p}(\Omega)$ and

$$\langle fh(u), \varphi \rangle_{\mathscr{D}'(\Omega), \mathscr{D}(\Omega)} = \langle f, \varphi h(u) \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)}.$$

The same type of identity gives a meaning in $\mathscr{D}'(\Omega)$ to the term $[-\operatorname{div}(a(x, u, Du))]\, h(u)$ since $-\operatorname{div}(a(x, u, Du))$ belongs to $W^{-1,p'}(\Omega)$.

Moreover since Φh and $\Phi h'$ belong to $(C_c^0(\mathbb{R}))^N$ (the class of continuous functions with compact support) the functions $\Phi(u)\, h(u)$ and $\Phi(u)\, h'(u)$ lie in $(L^\infty(\Omega))^N$ for any measurable function u and then

$$\operatorname{div}(\Phi(u)\, h(u)) \in W^{-1,\infty}(\Omega),$$

$$\Phi(u)\, h'(u)\, Du \in L^p(\Omega).$$

Remark 2. The definition of renormalized solution used here is nothing but the adaptation to the present elliptic setting of the idea of renormalization introduced by R. J. DiPerna and P.-L. Lions in their important papers [8, 9] dealing with the existence of a solution of the Boltzmann equation. Similar ideas also appear in the papers [1] by P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vazquez, and [11] by P.-L. Lions and F. Murat.

Theorem 1 as well as Theorems 2 and 3 below generalizes to the present setting of Leray–Lions operators the results we obtained in [5] for quasilinear operators ($p = 2$) and $g = 0$.

1.2. *Renormalized Solutions and Usual Weak Solutions*

A natural question now arises: Is any renormalized solution a usual weak solution of the original problem (1.4)–(1.5)?

To obtain such a result it seems to be natural to assume $\Phi(u)$ to belong to $(L^1_{\mathrm{loc}}(\Omega))^N$ in order to give a meaning to each term of (1.4), but even so we do not know the answer to the question, except when stronger assumptions are satisfied by u, as it is shown in Theorem 2.

THEOREM 2. *Assume that* (1.1), (1.2), (1.3), (1.6) *hold and that* Φ *belongs to* $(C^1(\mathbb{R}))^N$, *and let* u *be any (renormalized) solution of* (1.10)–(1.11) *which satisfies*

$$\Phi(u) \in (L^1_{\mathrm{loc}}(\Omega))^N. \tag{1.12}$$

Define for $i \in \{1, ..., N\}$

$$\omega_i(t) = \int_0^t |\Phi_i'(\tau)| \, d\tau$$

and assume moreover that

$$\omega_i(u) \in L^1_{\mathrm{loc}}(\Omega). \tag{1.13}$$

Then u *is a usual weak solution of the original problem* (1.4)–(1.5).

Remark 3. The result of Theorem 2 still holds true if (1.13) is replaced by the assumption

$$\psi_i(u) \in L^1_{\mathrm{loc}}(\Omega), \tag{1.14}$$

where ψ_i is defined by

$$\psi_i(t) = \int_0^t |\Phi_i(\tau)| \, d\tau.$$

In this case Φ does not need to belong to $(C^1(\mathbb{R}))^N$ but only to $(C^0(\mathbb{R}))^N$.

Remark 4. If (1.12) holds true, a sufficient condition for (1.13) to hold is to have

$$|\omega_i(t)| \leqslant \mathrm{const}[|\Phi_i(t)| + |t|^{p^*} + 1],$$

where $p^* = Np/(N-p)$ when $p < N$ and p^* is any positive number when $p = N$ (if $p > N$, $W_0^{1,p}(\Omega)$ is embedded in $L^\infty(\Omega)$ and anything is straightforward).

Another sufficient condition for (1.13) to hold true whenever (1.12) is satisfied is to assume each $\Phi_i(t)$ to be monotone (either nondecreasing or nonincreasing) when t is sufficiently large.

1.3. *Energy Identities for Renormalized Solutions*

Let us now multiply formally (1.4) by u and integrate by parts. We get

$$\int_\Omega a(x, u, Du)\, Du\, dx + \int_\Omega \Phi(u)\, Du\, dx + \int_\Omega g(x, u) u\, dx = \langle f, u \rangle.$$

Define $\tilde{\Phi} \in (C^1(\mathbb{R}))^N$ as $\tilde{\Phi}(t) = \int_0^t \Phi(\tau)\, d\tau$. Then, formally, $\operatorname{div}(\tilde{\Phi}(u)) = \Phi(u)\, Du$ and by the Divergence Theorem

$$\int_\Omega \Phi(u)\, Du\, dx = \int_\Omega \operatorname{div}(\tilde{\Phi}(u))\, dx = \int_{\partial\Omega} \tilde{\Phi}(0) n\, ds = 0$$

since $u = 0$ on $\partial\Omega$ and $\tilde{\Phi}(0) = 0$. Thus formally

$$\int_\Omega a(x, u, Du)\, Du\, dx + \int_\Omega g(x, u) u\, dx = \langle f, u \rangle. \tag{1.15}$$

Let us note that most of the operations performed before are purely formal. However, (1.15) (and even its extension (1.16)) can be proved to hold whenever u is a (renormalized) solution of (1.10)–(1.11). Indeed we have:

THEOREM 3. *Assume that* (1.1), (1.2), (1.3), (1.6), *and* (1.7) *hold and let* u *be any (renormalized) solution of* (1.10)–(1.11). *Then*

$$\int_\Omega s'(u)\, a(x, u, Du)\, Du\, dx + \int_\Omega g(x, u)\, s(u)\, dx = \langle f, s(u) \rangle_{W^{-1,p'}(\Omega),\, W_0^{1,p}(\Omega)} \tag{1.16}$$

for any Lipschitz continuous, piecewise $C^1(\mathbb{R})$ *function* s *such that* $s(0) = 0$.

Remark 5. Theorem 3 is particularly useful to prove L^s-regularity results for renormalized solutions since it justifies multiplications of the equation in renormalized form (1.10)–(1.11) by test functions which are nonlinear in u, for example, by $|T_n(u)|^r\, T_n(u)$, where T_n is the truncation to the level n. This is the usual way to obtain L^s-regularity results in equations of the form (1.4)–(1.5) when Φ is bounded and f has some $W^{-1,q}$ regularity ($q > p'$).

Theorem 3 will be used in Section 4 in order to prove L^s- and L^∞-regularity results for the renormalized solutions.

1.4. *Test Functions in Renormalization*

In the definition (1.10)–(1.11) of renormalized solutions, we used test functions of the form $h(u)\varphi$ where φ belongs to $\mathscr{D}(\Omega)$ and h to $C_c^1(\mathbb{R})$, u being the solution. It is actually possible to take φ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see (3.5) below) but a larger class of test functions can be used, as proved by the following result.

THEOREM 4. *Assume that* (1.1), (1.2), (1.3), (1.6), *and* (1.7) *hold and let u be any* (*renormalized*) *solution of* (1.10)–(1.11). *Then, for any* $w \in W_0^{1,p}(\Omega)$ *such that*

$$Dw = 0 \qquad \text{a.e. on } \{x : |u(x)| \geqslant k\} \text{ for some } k \in \mathbb{R}^+, \tag{1.17}$$

$$g(x, u)w \in L^1(\Omega), \tag{1.18}$$

we have

$$\begin{cases} \int_\Omega a(x, u, Du)\, Dw\, dx + \int_\Omega \bar{\Phi}_k(u)\, Dw\, dx + \int_\Omega g(x, u) w\, dx \\ \qquad = \langle f, w \rangle_{W^{-1,p'}(\Omega),\, W_0^{1,p}(\Omega)}, \end{cases} \tag{1.19}$$

where $\bar{\Phi}_k(t)$ is defined by

$$\bar{\Phi}_k(t) = \begin{cases} \Phi(t) & \text{if } \ |t| \leqslant k \\ \Phi\left(k \dfrac{t}{|t|}\right) & \text{if } \ |t| \geqslant k. \end{cases}$$

Note that $\bar{\Phi}_k$ belongs to $(C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$, which implies that $\bar{\Phi}_k(u)$ lies in $(L^\infty(\Omega))^N$ for any measurable function u.

1.5. *L^s-Regularity of Renormalized Solutions*

L^s- and L^∞-regularity results for solutions of (1.10)–(1.11) can be proved when some $W^{-1,q}$ regularity ($q > p'$) is assumed on the right hand side f. Let us recall that the first L^s-regularity results were proved by G. Stampacchia in [12] when the principal part A of the operator is linear and g and Φ are identically zero. These results were extended to the non-linear case when Φ is zero in [3] and next when Φ has polynomial growth in [4].

Denote

$$f = -\operatorname{div} c, \qquad \text{with} \quad c \text{ in } (L^q(\Omega))^N,\ p' \leqslant q \leqslant +\infty \tag{1.20}$$

and define the number r^* by

$$r^* = Nr/(N - r) \qquad \text{for} \quad 1 \leqslant r < N.$$

THEOREM 5. *Assume that* (1.1), (1.2), (1.3), (1.7), (1.8), *and* (1.20) *hold true and let u be any (renormalized) solution of* (1.10)–(1.11).

If $p' \leqslant q < N/(p-1)$, *then u belongs to* $L^{[q(p-1)]^*}(\Omega)$ *and*

$$\|u\|_{L^{[q(p-1)]^*}(\Omega)} \leqslant k_1 \|c\|^{1/(p-1)}_{(L^q(\Omega))^N}, \qquad k_1 > 0. \tag{1.21}$$

If $q > N/(p-1)$, *then u belongs to* $L^\infty(\Omega)$ *and*

$$\|u\|_{L^\infty(\Omega)} \leqslant k_2 \|c\|^{1/(p-1)}_{(L^q(\Omega))^N}, \qquad k_2 > 0. \tag{1.22}$$

If $q = N/(p-1)$, *then u belongs to* $L^s(\Omega)$ *for any* $s < +\infty$ *and*

$$\|u\|_{L^s(\Omega)} \leqslant k_3(s) \|c\|^{1/(p-1)}_{(L^q(\Omega))^N}, \qquad k_3(s) > 0. \tag{1.23}$$

More regularity can be obtained on the solutions of (1.10)–(1.11) when a further coerciveness condition is assumed on the term $g(x, u)$, that is

$$g(x, t)\, t \geqslant v\, |t|^r, \qquad \text{a.e. } x \text{ in } \Omega, \qquad \forall t \in \mathbb{R}, \tag{1.24}$$

for some $r \geqslant 1$ and $v > 0$. Then the following result can be proved.

THEOREM 6. *Assume that* (1.1), (1.2), (1.3), (1.7), (1.8), (1.20), *and* (1.24) *hold and let u be any (renormalized) solution of* (1.10)–(1.11). *If* $q \geqslant p'$, *then u belongs to* $L^{rq/p'}(\Omega)$ *and*

$$\|u\|_{L^{rq/p'}(\Omega)} \leqslant k_4 \|c\|^{p'/r}_{(L^q(\Omega))^N}, \qquad k_4 > 0. \tag{1.25}$$

Remark 6. When $q > N/(p-1)$ the result of Theorem 5 is stronger than (1.25) since u then belongs to $L^\infty(\Omega)$. In this case no advantage is taken of hypothesis (1.24). The same holds true when $q = N/(p-1)$.

In contrast with this observation, note that for $q < N/(p-1)$, we have $rq/p' \geqslant [q(p-1)]^*$ whenever $r \geqslant Np/(N - qp + q)$. In other words hypothesis (1.24) implies some gain of regularity when r is sufficiently large.

1.6. *Examples of Applications of the L^s-Regularity Results to the Existence of Usual Weak Solutions*

The regularity results obtained in Theorems 5 and 6 for renormalized solutions can be used, jointly with Theorems 1 and 2, to prove the existence of a (usual) weak solution for the initial problem (1.4)–(1.5), when f satisfies (1.20) while Φ satisfies

$$|\Phi(t)| \leqslant \text{const}(1 + |t|^\gamma) \qquad \text{for some} \quad \gamma \geqslant 0. \tag{1.26}$$

We give here two examples of possible applications.

PROPOSITION 1. *Assume that* (1.1), (1.2), (1.3), (1.7), (1.8), (1.9), (1.20), *and* (1.26) *hold true with* $p' \leqslant q < N/(p-1)$.

Then there exists a (*usual*) *weak solution of* (1.4)–(1.5) *whenever one of the following two conditions is satisfied*:

$$0 \leqslant \gamma \leqslant [q(p-1)]^* - 1 \tag{1.27}$$

$$\begin{cases} 0 \leqslant \gamma \leqslant [q(p-1)]^* \\ \omega(u) \in (L^1_{\mathrm{loc}}(\Omega))^N \qquad \textit{with} \quad \omega_i(t) = \int_0^t |\Phi_i'(\tau)| \, d\tau \textit{ for } i \in \{1, ..., N\}; \end{cases} \tag{1.28}$$

if (1.28) *is assumed to hold one has to replace* (1.7) *by the assumption that* Φ *belongs to* $(C^1(\mathbb{R}))^N$.

Remark 7. Let us compare the result of this proposition with the results obtained in [4]. Hypothesis (1.27) is less restrictive that the hypothesis $\gamma < [q(p-1)]^*/p'$ made in [4, Sect. 1].

Concerning (1.28) let us observe that the present result allows one to have $\gamma \leqslant [q(p-1)]^*$ if each $\Phi_i(t)$ is monotone at infinity (see Remark 4); moreover the existence result is here proved for any Leray–Lions operator while in [4, Sect. 4] a growth γ strictly less than $[q(p-1)]^*$ was allowed only in the case where A was a quasilinear operator $-\mathrm{div}(b(x, u)\, Du)$ with $p = 2$.

Remark Added in Proof. The recent paper [13] improves the result of Proposition 1 when $g = 0$. Indeed in this paper it is proved that there exists a (usual) weak solution of (1.4), (1.5) when $g = 0$ and (1.1), (1.2), (1.3), (1.7), (1.20), and (1.26) hold with $p' \leqslant q < N/(p-1)$ and

$$0 \leqslant \gamma \leqslant [q(p-1)]^*$$

(compare with (1.27) and (1.28)).

PROPOSITION 2. *Assume that* (1.1), (1.2), (1.3), (1.7), (1.8), (1.9), (1.20), (1.24), *and* (1.26) *hold with* $p' \leqslant q < N/(p-1)$ *and* $r \geqslant Np/(N - qp + q)$. *Then there exists a* (*usual*) *weak solution of* (1.4)–(1.5), *whenever one of the following conditions is satisfied*

$$0 \leqslant \gamma \leqslant (rq/p') - 1 \tag{1.29}$$

$$0 \leqslant \gamma \leqslant rq/p' \qquad \textit{and} \qquad \omega(u) \in (L^1_{\mathrm{loc}}(\Omega))^N; \tag{1.30}$$

if (1.30) *is assumed to hold one has to replace* (1.7) *by the assumption that* Φ *belongs to* $(C^1(\mathbb{R}))^N$.

Remark 8. In both Propositions 1 and 2 we limited ourselves to the case $q < N/(p-1)$, since any renormalized solution belongs to $L^\infty(\Omega)$ when $q > N/(p-1)$ and is thus a (usual) weak solution. (The same is true when $q = N/(p-1)$ and Φ has a polynomial growth.)

Similarly in Proposition 2 we limited ourselves to the case $r \geqslant Np/(N-qp+q)$ since otherwise no advantage is taken of hypothesis (1.24) (see Remark 6).

2. Proof of Theorem 1

Step 1. *Approximate Solutions and Weak Convergence*

Let us define for each $k > 0$, the truncation

$$T_k(t) = \begin{cases} t & \text{if} \quad |t| \leqslant k \\ k\,\dfrac{t}{|t|} & \text{if} \quad |t| \geqslant k \end{cases}$$

and, for each $\varepsilon > 0$, the approximations

$$\Phi_\varepsilon(t) = \Phi(T_{1/\varepsilon}(t)), \qquad g_\varepsilon(x, t) = T_{1/\varepsilon}(g(x, t)).$$

From well known results due to J. Leray and J.-L. Lions [10], the following nonlinear elliptic problem

$$\begin{cases} -\operatorname{div}(a(x, u_\varepsilon, Du_\varepsilon)) - \operatorname{div}(\Phi_\varepsilon(u_\varepsilon)) + g_\varepsilon(x, u_\varepsilon) = f & \text{in } \mathscr{D}'(\Omega) \\ u_\varepsilon \in W_0^{1,p}(\Omega) \end{cases} \tag{2.1}$$

has at least one solution u_ε.

Define $\tilde{\Phi}_\varepsilon(t) = \int_0^t \Phi_\varepsilon(\tau)\, d\tau$. By the Divergence Theorem (the use of which is licit here since $\tilde{\Phi}_\varepsilon(u_\varepsilon)$ belongs to $(W_0^{1,p}(\Omega))^N$) we get

$$\langle -\operatorname{div} \Phi_\varepsilon(u_\varepsilon), u_\varepsilon \rangle = \int_\Omega \Phi_\varepsilon(u_\varepsilon)\, Du_\varepsilon\, dx = \int_\Omega \operatorname{div} \tilde{\Phi}_\varepsilon(u_\varepsilon)\, dx$$

$$= \int_{\partial\Omega} \Phi_\varepsilon(0) n\, ds = 0.$$

If we multiply (2.1) by u_ε and integrate by parts, using the previous identity and the sign condition (1.8) we get

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon)\, Du_\varepsilon\, dx \leqslant \langle f, u_\varepsilon \rangle.$$

Now the coerciveness assumption (1.3) implies that

$$\|u_\varepsilon\|^{p-1}_{W_0^{1,p}(\Omega)} \leqslant \frac{1}{\alpha}\, \|f\|_{W^{-1,p'}(\Omega)} \tag{2.2}$$

$$\int_\Omega g(x, u_\varepsilon)\, u_\varepsilon\, dx \leqslant c_3. \tag{2.3}$$

So there exists a subsequence (again denoted by u_ε) and some $u \in W_0^{1,p}(\Omega)$ such that

$$u_\varepsilon \rightharpoonup u \qquad \text{weakly in } W_0^{1,p}(\Omega) \text{ and a.e. in } \Omega \text{ as } \varepsilon \to 0. \tag{2.4}$$

By the sign condition (1.8) and Fatou's Theorem we obtain from (2.3)

$$g(x, u)u \in L^1(\Omega). \tag{2.5}$$

On the other hand

$$g_\varepsilon(x, u_\varepsilon) \to g(x, u) \qquad \text{a.e. in } \Omega \text{ as } \varepsilon \to 0$$

while for any $n > 0$ and any measurable set E we have in view of (1.8), (1.9), (2.3)

$$\int_E |g_\varepsilon(x, u_\varepsilon)|\, dx \leqslant \int_{E \cap \{|u_\varepsilon| \leqslant n\}} |g_\varepsilon(x, u_\varepsilon)|\, dx + \frac{1}{n} \int_{E \cap \{|u_\varepsilon| \geqslant n\}} g_\varepsilon(x, u_\varepsilon)\, u_\varepsilon\, dx$$
$$\leqslant \int_E h_n(x)\, dx + \frac{1}{n}\, c_3.$$

Then, by Vitali's Theorem

$$g_\varepsilon(x, u_\varepsilon) \to g(x, u) \qquad \text{strongly in } L^1(\Omega). \tag{2.6}$$

Step 2. *Strong Convergence of* u_ε

Our aim is now to prove that u_ε strongly converges to u in $W_0^{1,p}(\Omega)$. This will allow us to pass to the limit in (2.1) after multiplication by $h(u_\varepsilon)$ and to complete the proof in Step 6 below. To prove the strong convergence result, let us recall the following lemma which is a variation of a result of J. Leray and J.-L. Lions (see [7, 6] for the proof).

LEMMA. *Assume that* (1.1), (1.2), (1.3) *hold true as well as*

$$u_\varepsilon \rightharpoonup u \qquad \text{weakly in } W_0^{1,p}(\Omega) \text{ and a.e. in } \Omega, \tag{2.7}$$

and that

$$\int_\Omega [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]\, D(u_\varepsilon - u)\, dx \to 0 \qquad \text{as} \quad \varepsilon \to 0. \tag{2.8}$$

Then

$$u_\varepsilon \to u \qquad \text{strongly in } W_0^{1,p}(\Omega). \tag{2.9}$$

In view of the lemma, we just need to prove (2.8) in order to obtain the desired result (2.9). This will be done in the following three steps of the proof.

Step 3. *First Estimate*

Define, for each $k > 0$ the set

$$E_k^\varepsilon = \{x \in \Omega : |u_\varepsilon(x)| \geqslant k\}.$$

Using in (2.1) the test function $v_\varepsilon = u_\varepsilon - T_k(u_\varepsilon)$ and the sign condition (1.8) on g we obtain in a way similar to that of Step 1 (see [5] if necessary) that for any fixed $k > 0$

$$\limsup_{\varepsilon \to 0} \int_{E_k^\varepsilon} |Du_\varepsilon|^p\, dx \leqslant \frac{1}{\alpha} \langle f, u - T_k(u) \rangle. \tag{2.10}$$

Step 4. *Second Estimate*

Define for $i, j > 0$

$$F_{ij}^\varepsilon = \{x \in \Omega : |u_\varepsilon(x) - T_j(u(x))| \leqslant i\}.$$

We will prove that

$$\begin{cases} \lim\limits_{\varepsilon \to 0} \displaystyle\int_{F_{ij}^\varepsilon} [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]\, D(u_\varepsilon - T_j(u))\, dx \\[2ex] \qquad = \langle f, T_i(u - T_j(u)) \rangle - \displaystyle\int_\Omega g(x, u)\, T_i(u - T_j(u))\, dx \\[2ex] \qquad\quad - \displaystyle\int_\Omega a(x, u, Du)\, D(T_i(u - T_j(u)))\, dx. \end{cases} \tag{2.11}$$

In order to obtain (2.11), let us use $w_\varepsilon = T_i(u_\varepsilon - T_j(u))$ as test function in (2.1) (see [2, 5], for the use of such nonlinear test functions):

$$\begin{cases} \int_\Omega a(x, u_\varepsilon, Du_\varepsilon)\, Dw_\varepsilon\, dx - \langle \operatorname{div} \Phi_\varepsilon(u_\varepsilon), w_\varepsilon \rangle \\ \qquad + \int_\Omega g_\varepsilon(x, u_\varepsilon)\, w_\varepsilon\, dx = \langle f, w_\varepsilon \rangle. \end{cases} \tag{2.12}$$

Let us first prove that for i, j fixed

$$\lim_{\varepsilon \to 0} \langle \operatorname{div} \Phi_\varepsilon(u_\varepsilon), w_\varepsilon \rangle = 0. \tag{2.13}$$

Indeed

$$Dw_\varepsilon = \begin{cases} D(u_\varepsilon - T_j(u)) & \text{on} \quad F_{ij}^\varepsilon \\ 0 & \text{on} \quad \Omega - F_{ij}^\varepsilon. \end{cases}$$

Since

$$|u_\varepsilon(x)| \leqslant |u_\varepsilon(x) - T_j(u(x))| + |T_j(u(x))| \leqslant i + j \qquad \text{on } F_{ij}^\varepsilon,$$

we have

$$\Phi_\varepsilon(u_\varepsilon(x)) = \Phi(T_{1/\varepsilon}(u_\varepsilon(x))) = \Phi(T_{i+j}(u(x))) \qquad \text{on} \quad F_{ij}^\varepsilon \qquad \text{for } \frac{1}{\varepsilon} \geqslant i + j.$$

Thus if $1/\varepsilon \geqslant i + j$

$$\langle -\operatorname{div} \Phi_\varepsilon(u_\varepsilon), w_\varepsilon \rangle = \int_\Omega \Phi_\varepsilon(u_\varepsilon)\, Dw_\varepsilon\, dx = \int_\Omega \Phi(T_{i+j}(u_\varepsilon))\, Dw_\varepsilon\, dx.$$

As $\varepsilon \to 0$, this term tends to

$$\int_\Omega \Phi(T_{i+j}(u))\, DT_i(u - T_j(u))\, dx$$

which vanishes using the Divergence Theorem as in the identity established after (2.1). This proves (2.13).

Let us now study the term $\int_\Omega g_\varepsilon(x, u_\varepsilon)\, w_\varepsilon\, dx$ of (2.12). We have

$$w_\varepsilon \overset{*}{\rightharpoonup} T_i(u - T_j(u)) \qquad \text{weakly * in } L^\infty(\Omega). \tag{2.14}$$

By (2.14), (2.6), we get

$$\lim_{\varepsilon \to 0} \int_\Omega g_\varepsilon(x, u_\varepsilon)\, w_\varepsilon\, dx = \int_\Omega g(x, u)\, T_i(u - T_j(u))\, dx.$$

Since w_ε tends to $T_i(u-T_j(u))$ weakly in $W_0^{1,p}(\Omega)$, we have proved that

$$\begin{cases} \lim_{\varepsilon \to 0} \int_\Omega a(x, u_\varepsilon, Du_\varepsilon)\, Dw_\varepsilon\, dx \\ \qquad = \langle f, T_i(u-T_j(u))\rangle - \int_\Omega g(x, u)\, T_i(u-T_j(u))\, dx. \end{cases} \tag{2.15}$$

By the strong convergence of u_ε to u in $L^p(\Omega)$ and (1.1), it is easy to obtain (2.11) from (2.15).

Step 5. *Proof of* (2.8)

In (2.8) let us split the integral over Ω as

$$\int_\Omega [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]\, D(u_\varepsilon - u)\, dx = I^\varepsilon_{ij} + II^\varepsilon_{ij} \tag{2.16}$$

where

$$I^\varepsilon_{ij} = \int_{F^\varepsilon_{ij}} [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]\, D(u_\varepsilon - u)\, dx$$

$$II^\varepsilon_{ij} = \int_{\Omega - F^\varepsilon_{ij}} [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]\, D(u_\varepsilon - u)\, dx.$$

Note that

$$\begin{aligned} I^\varepsilon_{ij} = &\int_{F^\varepsilon_{ij}} [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]\, D(u_\varepsilon - T_j(u))\, dx \\ &+ \int_{F^\varepsilon_{ij}} [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]\, D(T_j(u) - u)\, dx. \end{aligned}$$

Now observe that the right hand side of (2.11) is bounded by a function of j which goes to zero as j tends to ∞: indeed

$$\left| \int_\Omega a(x, u, Du)\, DT_i(u - T_j(u))\, dx \right| \leqslant \|a(x, u, Du)\|_{(L^{p'}(\Omega))^N}\, \|u - T_j(u)\|_{W_0^{1,p}(\Omega)}$$

$$|\langle f, T_i(u - T_j(u))\rangle| \leqslant \|f\|_{W^{-1,p'}(\Omega)}\, \|u - T_j(u)\|_{W_0^{1,p}(\Omega)}$$

$$\left| \int_\Omega g(x, u)\, T_i(u - T_j(u))\, dx \right| \leqslant \int_{\{|u| \geqslant j\}} g(x, u) u\, dx$$

and we know from (2.5) that $g(x, u)u$ belongs to $L^1(\Omega)$. This allows us to estimate the first term of I^ε_{ij}.

Using the boundedness of u_ε in $W_0^{1,p}(\Omega)$ and (1.1) we can estimate the second term of I_{ij}^ε in the same way. We thus deduce that

$$\limsup_{\varepsilon \to 0} I_{ij}^\varepsilon \leqslant F(j), \qquad \text{where} \quad F(j) \to 0 \text{ as } j \to +\infty. \tag{2.17}$$

Let us now come to II_{ij}^ε. Since for each $i > j > 0$

$$E_{i-j}^\varepsilon \supseteq \Omega - F_{ij}^\varepsilon$$

we have

$$\begin{cases} II_{ij}^\varepsilon \leqslant \displaystyle\int_{E_{i-j}^\varepsilon} [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]\, D(u_\varepsilon - u)\, dx \\ \quad \leqslant \displaystyle\int_{E_{i-j}^\varepsilon} [c_2\, |Du_\varepsilon|^{p-1} + c_2\, |Du|^{p-1} + 2c_1\, |u_\varepsilon|^{p-1} + 2d(x)] \\ \qquad\qquad [|Du_\varepsilon| + |Du|]\, dx \\ \quad \leqslant c_4 \left\{ \displaystyle\int_{E_{i-j}^\varepsilon} |Du_\varepsilon|^p\, dx + \int_{E_{i-j}^\varepsilon} [|Du|^p + |u_\varepsilon|^p + |d|^{p'}]\, dx \right\}. \end{cases} \tag{2.18}$$

By the estimate (2.10) the first term of the right hand side of (2.18) is small provided $i-j$ is sufficiently large. Moreover using Lebesgue's dominated convergence Theorem and Vitali's Theorem, we have

$$\limsup_{\varepsilon \to 0} \int_{E_{i-j}^\varepsilon} [|Du|^p + |u_\varepsilon|^p + |d|^{p'}]\, dx$$
$$\leqslant \int_{\{|u| \geqslant i-j\}} [|Du|^p + |u|^p + |d|^{p'}]\, dx$$

which is small again if $i-j$ is sufficiently large. Thus II_{ij}^ε is small if $i-j$ is large.

This last result, (2.17), and (2.16) prove that (2.8) holds true. By the lemma this implies (2.9).

Step 6. *End of the Proof*

Let us now multiply (2.1) by $h(u_\varepsilon)\varphi$, where $h \in C_c^1(\mathbb{R})$ and $\varphi \in \mathscr{D}(\Omega)$ and integrate by parts: we obtain

$$\begin{cases} \displaystyle\int_\Omega [a(x, u_\varepsilon, Du_\varepsilon) + \Phi_\varepsilon(u_\varepsilon)][h'(u_\varepsilon)\varphi\, Du_\varepsilon + h(u_\varepsilon)\, D\varphi]\, dx \\ \qquad + \displaystyle\int_\Omega g_\varepsilon(x, u_\varepsilon)\, h(u_\varepsilon)\varphi\, dx = \langle f, h(u_\varepsilon)\varphi \rangle. \end{cases} \tag{2.19}$$

Since h and h' have compact support on $\mathbb{R}$, we have for ε sufficiently small

$$\Phi_\varepsilon(t)\, h(t) = \Phi(T_{1/\varepsilon}(t))\, h(t) = \Phi(t)\, h(t)$$

$$\Phi_\varepsilon(t)\, h'(t) = \Phi(T_{1/\varepsilon}(t))\, h'(t) = \Phi(t)\, h'(t),$$

and the functions Φh and $\Phi h'$ belong to $(C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$. Now, using (2.6) and (2.9) it is easy to pass to the limit in each term of (2.19) and to obtain

$$\begin{cases} \int_\Omega a(x, u, Du)[h'(u)\varphi\, Du + h(u)\, D\varphi]\, dx \\ + \int_\Omega \Phi(u)\, h'(u)\varphi\, Du\, dx + \int_\Omega \Phi(u)\, h(u)\, D\varphi\, dx \\ + \int_\Omega g(x, u)\, h(u)\varphi\, dx = \langle f, h(u)\varphi\rangle \qquad \forall h \in C_c^1(\mathbb{R}),\ \forall \varphi \in \mathscr{D}(\Omega) \end{cases} \tag{2.20}$$

which is equivalent to (1.11). Theorem 1 is proved.

3. Proof of Theorems 2, 3, and 4

3.1. *Proof of Theorem* 2 *and Remark* 3

Let u be a solution of (1.10)–(1.11). Consider $H \in \mathscr{D}(\mathbb{R})$ such that

$$H(t) = 1 \text{ if } |t| \leqslant 1, \qquad H(t) = 0 \text{ if } |t| \geqslant 2,$$

$$|H(t)| \leqslant 1, \qquad |H'(t)| \leqslant 2\ \forall t \in \mathbb{R}$$

and define for $\varepsilon > 0$, h_ε by

$$h_\varepsilon(t) = \begin{cases} H\left(t + \frac{1}{\varepsilon}\right) & \text{if } t + \frac{1}{\varepsilon} \leqslant 0 \\ 1 & \text{if } |t| \leqslant \frac{1}{\varepsilon} \\ H\left(t - \frac{1}{\varepsilon}\right) & \text{if } t - \frac{1}{\varepsilon} \geqslant 0. \end{cases} \tag{3.1}$$

The function h_ε belongs to $C_c^1(\mathbb{R})$ and can be used in (1.11). Then for any φ in $\mathscr{D}(\Omega)$ we have

$$\begin{cases} \int_\Omega a(x, u, Du)[h_\varepsilon'(u)\varphi\, Du + h_\varepsilon(u)\, D\varphi]\, dx \\ + \int_\Omega \Phi(u)\, h_\varepsilon'(u)\varphi\, Du\, dx + \int_\Omega \Phi(u)\, h_\varepsilon(u)\, D\varphi\, dx \\ + \int_\Omega g(x, u)\, h_\varepsilon(u)\varphi\, dx = \langle f, h_\varepsilon(u)\varphi \rangle. \end{cases} \tag{3.2}$$

Since

$$|h_\varepsilon(t)| \leqslant 1, \qquad h_\varepsilon(t) \to 1 \text{ as } \varepsilon \to 0$$

$$|h_\varepsilon'(t)| \leqslant 2, \qquad h_\varepsilon'(t) \to 0 \text{ as } \varepsilon \to 0,$$

and since Lebesgue's dominated convergence Theorem implies that

$$h_\varepsilon(u) \to 1 \qquad \text{strongly in } W^{1,p}(\Omega),$$

we can pass to the limit in the first and the last term of (3.2).

In the third and in the fourth integrals we can use Lebesgue's dominated convergence theorem since by (1.12)

$$|\Phi_i(u)\, h_\varepsilon(u)| \leqslant |\Phi_i(u)| \in L^1_{\text{loc}}(\Omega) \qquad \text{for} \quad i \in \{1, ..., N\}$$

$$|g(x, u)\, h_\varepsilon(u)| \leqslant |g(x, u)| \in L^1(\Omega).$$

We have now to pass to the limit in the second integral of (3.2). We can write it as

$$\int_\Omega \Phi(u)\, h_\varepsilon'(u)\varphi\, Du\, dx = \int_\Omega \operatorname{div}(\tilde{\psi}_\varepsilon(u))\varphi\, dx = -\int_\Omega \tilde{\psi}_\varepsilon(u)\, D\varphi\, dx, \tag{3.3}$$

where

$$\begin{cases} \tilde{\psi}_\varepsilon(t) = (\tilde{\psi}_{1\varepsilon}(t), ..., \tilde{\psi}_{N\varepsilon}(t)), \\ \tilde{\psi}_{i\varepsilon}(t) = \int_0^t \Phi_i(\tau)\, h_\varepsilon'(\tau)\, d\tau, \qquad i \in \{i, ..., N\}. \end{cases} \tag{3.4}$$

Now since

$$\tilde{\psi}_{i\varepsilon}(t) \to 0 \qquad \text{as} \quad \varepsilon \to 0 \qquad \forall t \in \mathbb{R}$$

$$\tilde{\psi}_{i\varepsilon}(t) = -\int_0^t \Phi_i'(\tau)\, h_\varepsilon(\tau)\, d\tau + \Phi_i(t)\, h_\varepsilon(t) - \Phi_i(0)\, h_\varepsilon(0)$$

we have

$$|\tilde{\psi}_{i\varepsilon}(t)| \leqslant \int_0^t |\Phi_i'(\tau)|\, d\tau + |\Phi_i(t)| + |\Phi_i(0)| = \omega_i(t) + |\Phi_i(t)| + |\Phi_i(0)|.$$

Since by the hypotheses $\omega_i(u)$ and $\Phi_i(u)$ belong to $L^1_{\text{loc}}(\Omega)$, we have, by Lebesgue's Theorem

$$\tilde{\psi}_{i\varepsilon}(u) \to 0 \qquad \text{strongly in } L^1_{\text{loc}}(\Omega)$$

and the last expression of (3.3) tends to zero.

We proved that for any $\varphi \in \mathscr{D}(\Omega)$

$$\int_\Omega a(x, u, Du)\, D\varphi\, dx + \int_\Omega \Phi(u)\, D\varphi\, dx + \int_\Omega g(x, u)\varphi\, dx = \langle f, \varphi \rangle$$

which is equivalent to (1.4). Theorem 2 is proved

In the setting of Remark 3, the term (see (3.3))

$$\int_\Omega \Phi(u)\, h_\varepsilon'(u)\varphi\, Du\, dx = -\int_\Omega \tilde{\psi}_\varepsilon(u)\, D\varphi\, dx$$

(where $\tilde{\psi}_\varepsilon$ is defined by (3.4)) tends again to zero as ε tends to zero. Indeed since

$$|\tilde{\psi}_{i\varepsilon}(t)| \leqslant 2\psi_i(t),$$

$$\tilde{\psi}_{i\varepsilon}(t) \to 0 \qquad \text{as} \quad \varepsilon \to 0 \qquad \forall t \in \mathbb{R},$$

we have, by Lebesgue's Theorem,

$$\tilde{\psi}_{i\varepsilon}(u) \to 0 \qquad \text{strongly in } L^1_{\text{loc}}(\Omega)$$

whenever (1.14) holds true.

3.2. *Proof of Theorem* 3

Let u be a solution of (1.10)–(1.11) and s be a Lipschitz continuous piecewise C^1 function from $\mathbb{R}$ to $\mathbb{R}$ such that $s(0) = 0$.

Step 1. Equation (1.11) is understood in the distributional sense and is thus equivalent to (2.20). We approximate any v of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ by a sequence φ_n of $\mathscr{D}(\Omega)$ such that (φ_n) is bounded in $L^\infty(\Omega)$, and φ_n strongly converges to v in $W_0^{1,p}(\Omega)$. Then (2.20) implies that

$$\begin{cases} \displaystyle\int_\Omega a(x, u, Du)[h'(u) v\, Du + h(u)\, Dv]\, dx \\ \displaystyle + \int_\Omega \Phi(u)\, h'(u) v\, Du\, dx + \int_\Omega \Phi(u)\, h(u)\, Dv\, dx + \int_\Omega g(x, u)\, h(u) v\, dx \\ = \langle f, h(u) v \rangle \qquad \forall h \in C_c^1(\mathbb{R}), \qquad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{3.5}$$

Step 2. We assume in this step that s is bounded.

Then $s(u) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and can be used as test function in (3.5) yielding

$$\int_\Omega a(x, u, Du)[h'(u)\, s(u) + h(u)\, s'(u)]\, Du\, dx$$
$$+ \int_\Omega \Phi(u)\, h'(u)\, s(u)\, Du\, dx + \int_\Omega \Phi(u)\, h(u)\, s'(u)\, Du\, dx$$
$$+ \int_\Omega g(x, u)\, h(u)\, s(u)\, dx = \langle f, h(u)\, s(u) \rangle \qquad \forall h \in C_c^1(\mathbb{R}).$$

Define now $v_1(t) = \int_0^t \Phi(\tau)\, h'(\tau)\, s(\tau)\, d\tau$. This function is $C^1(\mathbb{R})$ with bounded derivative in $\mathbb{R}$. Then we can apply the Divergence Theorem and get

$$\int_\Omega \Phi(u)\, h'(u)\, s(u)\, Du\, dx = \int_\Omega \operatorname{div}(v_1(u))\, dx = 0.$$

Similarly

$$\int_\Omega \Phi(u)\, h(u)\, s'(u)\, Du\, dx = \int_\Omega \operatorname{div}(v_2(u))\, dx = 0,$$

where

$$v_2(t) = \int_0^t \Phi(\tau)\, h(\tau)\, s'(\tau)\, d\tau.$$

We thus have

$$\begin{cases} \displaystyle\int_\Omega a(x, u, Du)[h'(u)\, s(u) + h(u)\, s'(u)]\, Du\, dx + \int_\Omega g(x, u)\, h(u)\, s(u)\, dx \\ = \langle f, h(u)\, s(u) \rangle, \qquad \forall h \in C_c^1(\mathbb{R}). \end{cases} \tag{3.6}$$

Use now in (3.6) the function h_ε defined in (3.1). It is easy to prove that, as $\varepsilon \to 0$

$$h_\varepsilon(u)\, s(u) \to s(u) \qquad \text{strongly in } W_0^{1,p}(\Omega)$$
$$g(x, u)\, h_\varepsilon(u)\, s(u) \to g(x, u)\, s(u) \qquad \text{strongly in } L^1(\Omega)$$

for any bounded, Lipschitz continuous, piecewise C^1 function s such that $s(0) = 0$. Passing to the limit in (3.6) proves (1.16) when s is bounded.

Step 3. The general case, where the function s is no more assumed to be bounded, is achieved using in (1.16) the test function $s_n(t) = T_n(s(t))$. This gives

$$\int_\Omega s_n'(u)\, a(x, u, Du)\, Du\, dx + \int_\Omega g(x, u)\, s_n(u)\, dx = \langle f, s_n(u) \rangle. \tag{3.7}$$

Since as $n \to \infty$

$$s_n(u) \to s(u) \qquad \text{strongly in } W_0^{1,p}(\Omega)$$
$$g(x, u)\, s_n(u) \to g(x, u)\, s(u) \qquad \text{strongly in } L^1(\Omega)$$

(recall that $g(x, u)u$ belongs to $L^1(\Omega)$ and that $|s(t)| \leqslant C\,|t|$, for some constant C, because s is a Lipschitz continuous function), we easily pass to the limit in (3.7) obtaining (1.16).

Theorem 3 is proved.

3.3. *Proof of Theorem* 4

Step 1. Consider first the case where, further to (1.17), the function w belongs to $L^\infty(\Omega)$ and note that (1.18) holds true in this case.

By Step 1 of the proof of Theorem 3, we can use w as test function in (3.5). With $h_\varepsilon \in C_c^1(\mathbb{R})$ given by (3.1) this yields

$$\begin{cases} \displaystyle\int_\Omega a(x, u, Du)[h_\varepsilon'(u) w\, Du + h_\varepsilon(u)\, Dw]\, dx \\ \displaystyle + \int_\Omega \Phi(u)\, h_\varepsilon'(u) w\, Du\, dx + \int_\Omega \Phi(u)\, h_\varepsilon(u)\, Dw\, dx \\ \displaystyle + \int_\Omega g(x, u)\, h_\varepsilon(u) w\, dx = \langle f, h_\varepsilon(u) w \rangle. \end{cases} \tag{3.8}$$

Because of (1.17) and of the definition of $\bar{\Phi}_k$ given after (1.19) we have on the first hand

$$\int_\Omega \Phi(u)\, h_\varepsilon(u)\, Dw\, dx = \int_\Omega \bar{\Phi}_k(u)\, Dw\, dx \qquad \text{if } \frac{1}{\varepsilon} \geqslant k,$$

and, on the other hand,

$$\int_\Omega \Phi(u)\, h'_\varepsilon(u) w\, Du = \int_\Omega \operatorname{div}(\tilde{\psi}_\varepsilon(u)) w\, dx$$
$$= -\int_\Omega \tilde{\psi}_\varepsilon(u)\, Dw\, dx = 0 \qquad \text{if } \frac{1}{\varepsilon} \geqslant k$$

(where $\tilde{\psi}_\varepsilon(t)$ is defined by (3.4) and thus satisfies $\tilde{\psi}_\varepsilon(t) = 0$ when $|t| \leqslant \frac{1}{\varepsilon}$).

Passing to the limit in (3.8), for ε tending to zero, implies that

$$\begin{cases} \int_\Omega a(x, u, Du)\, Dw\, dx + \int_\Omega \bar{\Phi}_k(u)\, Dw\, dx \\ \qquad + \int_\Omega g(x, u) w\, dx = \langle f, w \rangle \end{cases} \tag{3.9}$$

for any w in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ which satisfies (1.17).

Step 2. To obtain (1.19) for any w in $W_0^{1,p}(\Omega)$ satisfying (1.17) and (1.18), it is now sufficient to take $w_n = T_n(w)$ in (3.9) and to pass to the limit with n tending to infinity: note that w_n still satisfies (1.17) and that

$$|g(x, u)\, T_n(w)| \leqslant |g(x, u) w| \qquad \text{a.e. in } \Omega.$$

4. Proof of Theorems 5 and 6 and of Propositions 1 and 2

4.1. *Proof of Theorem* 5

Let us first consider the case $q = p'$. Then $q(p-1) = p$ and (1.21) immediately follows from (1.16) with $s(u) = u$.

Consider now the case $p' < q < N/(p-1)$. Using in (1.16) the Lipschitz continuous, piecewise C^1 function $s(t) = |T_n(t)|^m\, T_n(t)$ where T_n is the truncation at level n and $m = N(qp - q - p)/(N - qp + q)$ yields

$$\int_\Omega s'(u)\, a(x, u, Du)\, Du\, dx + \int_\Omega g(x, u)\, s(u)\, dx = \int_\Omega c(x)\, s'(u)\, Du\, dx. \tag{4.1}$$

From this relation, (1.3), (1.8), and Young's inequality we have

$$\begin{cases} \alpha \int_\Omega |DT_n(u)|^p \, |T_n(u)|^m \, dx \\ \leqslant \int_\Omega |c(x)| \, |T_n(u)|^m \, |DT_n(u)| \, dx \\ \leqslant \frac{\alpha}{2} \int_\Omega |DT_n(u)|^p \, |T_n(u)|^m \, dx + c_5 \int_\Omega |c(x)|^{p'} \, |T_n(u)|^m \, dx. \end{cases} \tag{4.2}$$

Then

$$\begin{cases} \int_\Omega (|DT_n(u)| \, |T_n(u)|^{m/p})^p \, dx \\ \leqslant \frac{2}{\alpha} c_5 \, \|c\|^{p'}_{(L^q(\Omega))^N} \left(\int_\Omega |T_n(u)|^{mq/(q-p')} \, dx \right)^{(q-p')/q}. \end{cases} \tag{4.3}$$

Now

$$(m/p + 1) \, D(T_n(u)) \, |T_n(u)|^{m/p} = D(|T_n(u)|^{m/p} \, T_n(u)).$$

By the Sobolev inequality, we obtain from (4.3) (since $p < N$ here)

$$\left(\int_\Omega (|T_n(u)|^{m/p+1})^{p^*} \right)^{p/p^*} \leqslant c_6 \, \|c\|^{p'}_{(L^q(\Omega))^N} \left(\int_\Omega |T_n(u)|^{mq/(q-p')} \, dx \right)^{(q-p')/q}.$$

By the choice of m we get (1.21) for $T_n(u)$ since $(m/p + 1) \, p^* = mq/(q-p') = [q(p-1)]^*$.

Passing to the limit on n implies (1.21) for u itself.

If $q = N/(p-1)$, then $c(x) \in (L^{\tilde{q}}(\Omega))^N$ for any $\tilde{q}$, $p' < \tilde{q} < N/(p-1)$, and the result (1.23) follows from (1.21).

Finally if $q > N/(p-1)$, the proof of the L^∞-estimate (1.22) is quite similar to the one given in [12, 3]. Indeed we use in (1.16) the function $s(t) = t - T_k(t)$, $k > 0$, and then we proceed as in [3] since now the term involving $\Phi(u)$ has disappeared.

4.2. *Proof of Theorem* 6

The proof is similar to the proof of Theorem 5. Consider the case $p' < q$ (the case $p' = q$ corresponds to $\sigma = 0$ in what follows) and use in (1.16) the function $s(t) = |T_n(t)|^\sigma \, T_n(t)$ with $\sigma = r(q-p')/p'$.

In view of (1.24) the term involving $g(x, u)$ gives a positive contribution to the estimate since $r \geqslant 1$ implies

$$\int_\Omega g(x, u) \, s(u) \, dx \geqslant \nu \int_\Omega |T_n(u)|^{r+\sigma} \, dx.$$

We then have in place of (4.2)

$$\alpha \int_\Omega |DT_n(u)|^p \, |T_n(u)|^\sigma \, dx + \frac{\nu}{\sigma+1} \int_\Omega |T_n(u)|^{r+\sigma} \, dx$$

$$\leqslant \int_\Omega |c(x)| \, |T_n(u)|^\sigma \, |DT_n(u)| \, dx$$

$$\leqslant \frac{\alpha}{2} \int |DT_n(u)|^p \, |T_n(u)|^\sigma \, dx + c_7 \int_\Omega |c(x)|^{p'} \, |T_n(u)|^\sigma \, dx.$$

Then, since $p' < q$ we have

$$\int_\Omega |T_n(u)|^{r+\sigma} \, dx \leqslant c_8 \int_\Omega |c(x)|^q \, dx + \frac{1}{2} \int_\Omega |T_n(u)|^{\sigma q/(q-p')} \, dx.$$

Since $r + \sigma = \sigma q/(q-p') = rq/p'$, the estimate (1.24) is obtained for $T_n(u)$. Passing to the limit on n implies (1.25) for u itself.

4.3. *Proof of Propositions* 1 *and* 2

Proof of Proposition 1. Under the hypotheses of Proposition 1, Theorem 1 applies and provides the existence of a renormalized solution. Actually this renormalized solution turns out to be a usual weak solution: indeed if (1.27) holds true, the estimate (1.21) of Theorem 5 ensures that both conditions (1.12) and (1.14) of Remark 3 are satisfied; if (1.28) holds true, the same estimate (1.21) and Theorem 2 imply the result since conditions (1.12) and (1.13) are satisfied.

Proof of Proposition 2. The proof is similar to the proof of Proposition 1. In this case one uses Theorem 6 to obtain L^s-regularity results sufficient to imply (1.12) and (1.14) (or (1.13)).

References

1. P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vazquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, to appear.
2. A. Bensoussan, L. Boccardo, and F. Murat, On a nonlinear partial differential equation having natural growth terms and unbounded solution, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **5**, No. 4 (1988), 347–364.
3. L. Boccardo and D. Giachetti, Alcune osservazioni sulla regolarità delle soluzioni di problemi fortemente non lineari e applicazioni, *Ricerche Mat.* **34**, No. 2 (1985), 309–323.
4. L. Boccardo and D. Giachetti, Existence results via regularity for some nonlinear elliptic problems, *Comm. Partial Differential Equations* **14**, No. 5 (1989), 663–680.

5. L. BOCCARDO, J. I. DIAZ, D. GIACHETTI, AND F. MURAT, Existence of a solution for a weaker form of a nonlinear elliptic equation, *in* "Recent Advances in Nonlinear Elliptic and Parabolic Problems (Proceedings, Nancy, 1988)" (P. Bénilan, M. Chipot, L. C. Evans, and M. Pierre, Eds.), pp. 229–246, Pitman Research Notes in Mathematics Series, Vol. 208, Longman, Harlow, 1989.
6. L. BOCCARDO, F. MURAT, AND J. P. PUEL, Existence of bounded solutions for nonlinear elliptic unilateral problems, *Ann. Mat. Pura Appl.* **152** (1988), 183–186.
7. F. E. BROWDER, Existence theorems for nonlinear partial differential equations, *in* "Proceedings of Symposia in Pure Mathematics" (S. S. Chern and S. Smale, Eds.), Vol. 16, pp. 1–60, Amer. Math. Soc., Providence, RI, 1970.
8. R. J. DIPERNA AND P.-L. LIONS, Global existence for the Fokker–Planck–Boltzmann equations, *Comm. Pure Appl. Math.* **11**, No. 2 (1989), 729–758.
9. R. J. DIPERNA AND P.-L. LIONS, On the Cauchy problem for Boltzmann equations: Global existence and weak stability, *Ann. of Math.* **130** (1989), 321–366.
10. J. LERAY AND J.-L. LIONS, Quelques résultats de Visik sur les problèmes elliptiques non linéaires par la méthode de Minty–Browder, *Bull. Soc. Math. France* **93** (1965), 97–107.
11. P.-L. LIONS AND F. MURAT, Sur les solutions renormalisées d'équations elliptiques non linéaires, *C. R. Acad. Sci. Paris*, and article, to appear.
12. G. STAMPACCHIA, Equations elliptiques du second ordre à coefficients discontinus, Séminaires de l'Université de Montréal, Vol. 16, Montréal, 1966.
13. L. BOCCARDO AND F. MURAT, An existence result via L^s-regularity for some nonlinear elliptic equations, *in* "Nonlinear Diffusion Equations and Their Equilibrium States, Volume III" (N. G. Llyod, W. N. Ni, L. A. Peletier, and J. Serrin, Eds.), pp. 145–152, Progress in Nonlinear Differential Equations and Their Applications, Vol. 8, Birkhäuser, Boston, 1992.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium