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The Propagation of the Free Boundary of the Solution of the Dam Problem and Related Problems

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Abstract We study the support of the solution of the dam problem and estimate its growth from above and from below for small t. We derive sufficient conditions in order that it grows with finite or infinite speed. Moreover a monotonicity result for the mushy region is proved.

KEY WORDS: Speed of propagation, free boundary, porous media,

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1. INTRODUCTION

It is well known that the solution of the Cauchy problem

$$(1.1) \quad \partial_t\beta(u) - \operatorname{div}(\nabla u + \mathrm{b}(u)) = 0 \qquad \text{in } \mathbb{R}^n \times]0,\infty[$$

$$(1.2) \quad u(\cdot,0) = u_0 \qquad \text{in } \mathbb{R}^n$$

has a compact support for every $t > 0$, provided that u_0 has the same property and β and b are properly chosen. This is the case for instance if

$$\beta(u) \sim u^{\varepsilon} \quad \text{as } u \to 0 \qquad \text{with} \qquad 0 < \varepsilon < 1,$$

which corresponds to a lack of parabolicity (more relevant if ε is very small), or if the convection term is very important (see e.g. [4], [5], [18], [31] and their references).

Here we deal with equations that can be seen also as limit cases of (1.1), for instance

$$(1.3)\quad \begin{cases} \partial_t \chi - \operatorname{div}(\nabla u + \chi e) = 0 \\ \chi \in H(u), \end{cases}$$

where e is a fixed vector and H is the maximal monotone graph associated to Heaviside function. Indeed (1.3) follows formally from (1.1) taking $\beta(u) = u^\varepsilon$ and $b(u) = u^\varepsilon e$, and letting $\varepsilon \to 0$ (see [9], [12], and their references).

More precisely we consider either the local behaviour of the solution of (1.3) or global properties in the case of a bounded domain $\Omega \subset \mathbb{R}^n$, with suitable boundary conditions.

If $-e$ is the gravity field and u is understood to be the pressure of an incompressible fluid in a homogeneous porous medium, (1.3) with suitable boundary and initial conditions is the well known dam problem (see [1], [6], [7], [11], [13], [14], [22], [29], [23], [30], [32], and their references); however similar equations describe a variety of physical situations: for instance the introduction of variable (smooth) coefficients gives both the nonhomogeneous dam problem and a model for lubrication problems, namely

$$\partial_t(hu) - \operatorname{div}(h^3\nabla u - h\chi e) = 0,$$

where e is a fixed vector and h is a strictly positive given function; taking instead $e = 0$ we obtain a model for Hele–Shaw flows (see [8], [20], [16], [21], and their references).

In this paper we consider only, for simplicity, the equations (1.3), which give both flows in porous media and Hele–Shaw flows according to $e \neq 0$ or $e = 0$; however our results can be taken as starting point for generalizations to the case of variable coefficients, which will include both the lubrication problem and the dam problem in nonhomogeneous domains. In the last case, however, not all results can be generalized: for instance it is well known that a mushy region can arise spontaneously if coefficients can jump.

The plan of the paper is the following: in section 1 we state the problem more carefully and give the basic definitions and results that we will use later; in section 2 we give criteria for piece-wise smooth functions to be either solutions or super- or subsolutions; in section 3 and 4 we estimate the growth of the support of the solution respectively from above and from below and show the existence or the non existence of a waiting time, that is a time t^* such that the flow moves just by gravity in $]0, t^*[$.

These results say essentially that the saturated region grows with finite speed provided it is produced by the given boundary pressure; on the contrary in some situations the saturated region generates instantaneously a strictly positive pressure.

In the last section we recall our local estimates and deduce global statements on the support of the solution and on the mushy region.

This paper completes the research begun by two of the authors in [15], where the speed of propagation was studied for interior points.

2. STATEMENT OF THE PROBLEM

In the following Ω is an open set in $\mathbb{R}^n$ with Lipschitz boundary $\Gamma = \partial\Omega$. For global results we shall need Ω to be bounded and for some local results we shall assume that Γ is smoother close to points of interest.

In the dam problem Ω represents a homogeneous isotropic porous medium and its boundary Γ is split into two relatively open subsets: the pervious part Γ_D and the impervious one Γ_N.

For simplicity we assume the following two conditions

$$(2.1)\quad \nu \cdot \mathrm{e} \leq 0 \quad \text{on } \Gamma_N \qquad \text{and} \qquad \nu \cdot \mathrm{e} \geq 0 \quad \text{on } \Gamma_D,$$

although the second one could be avoided, where ν is the outer normal unit vector on $\partial\Omega$. In the dam problem (2.1) mean that the bottom of Ω is impervious and the top is pervious; if $\mathrm{e} = 0$ (2.1) is empty.

Given two functions g and χ_0, on $\partial\Omega \times \left]0,\infty\right[$ and on Ω respectively, we define what we call a (weak) solution, assuming Ω to be bounded. For the unbounded case some obvious modifications have to be made (see [26]).

Definition 2.1. *A solution is a pair (u, χ) satisfying for every $T > 0$*

$$(2.2)\quad u \in L^2(0,T;\, H^1(\Omega)), \quad \chi \in L^\infty\left(\Omega \times \left]0,T\right[\right)$$

$$(2.3)\quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(1-\chi) = 0$$

$$(2.4)\quad u = g \qquad \text{on } \Gamma_D \times \left]0,T\right[$$

$$(2.5)\quad \int_{\Omega\times]0,T[} \left(-\chi\partial_t v + (\nabla u + \chi \mathrm{e}) \cdot \nabla v\right)\, dx\, dt \leq 0$$

for every $v \in H^1\left(\Omega \times \left]0,T\right[\right)$ such that

$$v(\cdot,0) = v(\cdot,T) = 0 \quad \text{in } \Omega$$

$$v \geq 0 \quad \text{on } \Gamma_D \times \left]0,T\right[, \qquad v = 0 \quad \text{on } \left(\Gamma_D \times \left]0,T\right[\right) \cap \{g > 0\}.$$

$$(2.6)\quad \chi(\cdot,0) = \chi_0. \;\square$$

Note that (2.6) makes sense, as (2.2) and (2.5) imply e.g.

$$\partial_t \chi \in L^2(0,T;\, H^{-1}(\Omega)). \;\square$$

In [15] it is proved that

(2.7) $\chi \in C^0([0,T]; L^p(\Omega)) \quad \forall p \in [1,\infty[$

(2.8) $\Delta u \geq 0 \quad$ in $\Omega \times]0,T[$

(2.9) $\partial_t \chi - \mathrm{div}(\chi e) \geq 0 \quad$ in $\Omega \times]0,T[$. □

Now we define (weak) super- and subsolutions.

Definition 2.2. *A supersolution is a pair (u,χ) satisfying (2.2), (2.3), (2.7), (2.9), and moreover*

(2.10) $u \geq g \quad$ *on* $\Gamma_D \times]0,T[$

(2.11) $$\int_{\Omega\times]0,T[} (-\chi\partial_t v + (\nabla u + \chi e)\cdot \nabla v)\, dx\, dt \geq 0$$

for every $v \in H^1(\Omega \times]0,T[)$ *such that*

$v(\cdot,0) = v(\cdot,T) = 0 \quad$ *in* Ω

$v \geq 0 \quad$ *in* $\Omega \times]0,T[, \qquad v = 0 \quad$ *on* $(\Gamma_D \times]0,T[)$

(2.12) $\chi(\cdot,0) \geq \chi_0$. □

Definition 2.3. *A subsolution is a pair (u,χ) satisfying (2.2), (2.3), (2.7), (2.9), and moreover*

(2.13) $u \leq g \quad$ *on* $\Gamma_D \times]0,T[$

(2.14) $$\int_{\Omega\times]0,T[} (-\chi\partial_t v + (\nabla u + \chi e)\cdot \nabla v)\, dx\, dt \leq 0$$

for every $v \in H^1(\Omega \times]0,T[)$ *such that*

$v(\cdot,0) = v(\cdot,T) = 0 \quad$ *in* Ω

$v \geq 0 \quad$ *in* $\Omega \times]0,T[, \qquad v = 0 \quad$ *on* $(\Gamma_D \times]0,T[) \cap \{u > 0\}$

(2.15) $\chi(\cdot,0) \leq \chi_0$. □

Note that solutions and subsolutions satisfy the so called *overflow condition*

(2.16) $(\nabla u + \chi e)\cdot \nu \leq 0 \quad$ on $(\Gamma_D \times]0,T[) \cap \{u = 0\}$

in a weak form, since test functions are just nonnegative on that part of the boundary (in the case $e = 0$ this condition on test functions would not be necessary and (2.16) would hold in any case just because $u \geq 0$ in $\Omega \times]0,T[$). □

In orded to use [15] we need further assumptions

(2.17) $\mathcal{D}\left(\overline{\Omega}\setminus\left(\overline{\Gamma}_D\cap\overline{\Gamma}_N\right)\right)$ is dense in $H^1(\Omega)$

(2.18) $g\in C^{0,1}\left(\overline{\Omega}\times[0,T]\right),\quad g\geq 0$

(2.19) $\mathcal{D}\left(\left(\overline{\Omega}\times[0,T]\right)\setminus\left(\overline{\Gamma}_D\times\,]0,T[\right)\cap\overline{\{g>0\}}\cap\{g=0\}\right)$

is dense in $H^1\left(\Omega\times\,]0,T[\right)$.

Moreover the condition obtained replacing g by u in (2.19) is fulfilled in the case of a subsolution u. □

Notice that (2.17) and (2.19) hold if $\overline{\Gamma}_D$ and $\operatorname{supp}\left(g\,|_{\Gamma_D\times]0,T[}\right)$ are smooth submanifolds.

From [13] one checks that the following basic statement holds under the provious assumptions

Comparison theorem 2.4. *Assume $(\underline{u},\underline{\chi})$ and $(\overline{u},\overline{\chi})$ are a sub– and a supersolution respectively. Then*

$$\underline{u}\leq\overline{u}\quad\text{and}\quad\underline{\chi}\leq\overline{\chi}\qquad\text{in }\Omega\times\,]0,T[.\ \square$$

It is clear that this statement holds if the initial time is any $t_0\geq 0$, instead of 0, and that any solution is both a sub– and a supersolution.

3. USEFUL CRITERIA ON THE FREE BOUNDARY

In this section we give sufficient conditions fo (u,χ) to be a solution, or a supersolution, or a subsolution, assuming that the set where $u>0$ is defined by an inequality of the form $F(x,t)>0$ (for some smooth F) and that u and χ are smooth both in $\{F>0\}$ and in $\{F<0\}$. Such pairs (u,χ) could be called strong solutions, or supersolutions, or subsolutions. More precisely we assume that

(3.1) $F\in C^1\left(\overline{\Omega}\times\,]0,T[\right)$ and $|F|+|\nabla F|+|\partial_t F|\neq 0$ in $\overline{\Omega}\times\,]0,T[$

and introduce the notations

$$(3.2)\quad \begin{cases} Q=\Omega\times\,]0,T[,\quad Q^+=Q\cap\{F>0\},\quad Q^-=Q\cap\{F<0\},\\ \mathcal{F}=Q\cap\{F=0\}.\end{cases}$$

Proposition 3.1. *Assume*

$$\chi_- \in C^0\left(\overline{Q^-}\right) \quad \text{and} \quad \partial_t \chi_- - \operatorname{div}(\chi_- \mathrm{e}) \in L^2\left(Q^-\right)$$

and define

$$\chi = \chi_- \quad in \;\; Q^- \qquad \text{and} \qquad \chi = 1 \quad in \;\; Q^+.$$

Then a sufficient condition for (2.9) *is*

$$\begin{aligned} &\partial_t \chi_- - \operatorname{div}(\chi_- \mathrm{e}) \geq 0 && in \;\; Q^- \\ &(1-\chi_-)\left(\partial_t F - \nabla F \cdot \mathrm{e}\right) \geq 0 && on \;\; \mathcal{F}. \; \square \end{aligned}$$

Proof. Let us denote by $\tilde{\nabla}$ the gradient operator in $n+1$ variables and by $\tilde{\nu}$ the outer normal unit vector on $Q \cap \partial Q^+$ with respect to Q^+.

Then $\tilde{\nu} = -\tilde{\nabla} F / |\tilde{\nabla} F|$ and integrations by parts over Q^+ and Q^- give for any nonnegative $v \in \mathcal{D}(Q)$

$$\iint_Q \chi\left(\partial_t v - \chi \nabla v \cdot \mathrm{e}\right)\, dx\, dt =$$

$$= -\iint_{Q^-} \left(\partial_t \chi - \operatorname{div}(\chi_- \mathrm{e})\right) v \, dx\, dt - \iint_{\mathcal{F}} (1-\chi_-)\, v \, \frac{\partial_t F - \nabla F \cdot \mathrm{e}}{|\tilde{\nabla} F|}\, d\sigma\, dt. \; \square$$

Again an integration by parts gives the following

Proposition 3.2. *Let* u_+ *and* χ_- *satisfy*

$$u_+ \in L^2(Q^+) \cap C^2\left(\overline{\Omega} \times \,]0,T[\, \cap \overline{Q^+}\right), \qquad \nabla u_+ \in \left(L^2(Q^+)\right)^n$$

$$\chi_- \in C^0\left(\overline{Q^-}\right), \qquad \partial_t \chi_- - \operatorname{div}(\chi_- \mathrm{e}) \in L^2(Q^-)$$

and define

$$u = \begin{cases} u_+ & in \;\; Q^+ \\ 0 & in \;\; Q^- \end{cases} \qquad \text{and} \qquad \chi = \begin{cases} 1 & in \;\; Q^+ \\ \chi_- & in \;\; Q^-. \end{cases}$$

Then, for every $v \in H^1(Q)$ *vanishing near* $t=0$ *and* $t=T$, *we have*

$$\iint \left(-\chi \partial_t v + (\nabla u + \chi \mathrm{e}) \cdot \nabla v\right)\, dx\, dt =$$

$$= -\iint_{Q^+} (\Delta u) v \, dx\, dt + \iint_{Q^-} \left(\partial_t \chi - \operatorname{div}(\chi_- \mathrm{e})\right) v \, dx\, dt +$$

$$+ \iint_{\mathcal{F}} \left((1-\chi_-)(\partial_t F - \nabla F \cdot \mathrm{e}) - \nabla u_+ \cdot \nabla F\right) \frac{v}{|\tilde{\nabla} F|}\, dx\, dt +$$

$$+ \iint_{\Gamma \times]0,T[\cap \partial Q^+} (\nabla u_+ + \mathrm{e}) \cdot \nu v \, d\sigma\, dt + \iint_{\Gamma \times]0,T[\cap \partial Q^-} \chi_- \mathrm{e} \cdot \nu v \, d\sigma\, dt. \; \square$$

Remark 3.3. From 3.2 one derives immediately sufficient conditions fo (u, χ) to satisfy (2.5), or (2.11), or (2.14). For instance (2.5) holds provided

$$\begin{array}{ll} \Delta u_+ = 0 & \text{in } Q^+ \\ \partial_t \chi_- - \operatorname{div}(\chi_- \mathrm{e}) = 0 & \text{in } Q_- \\ (1-\chi)(\partial_t F - \nabla F \cdot \mathrm{e}) & \\ \qquad - \nabla u_+ \cdot \nabla F = 0 & \text{on } \mathcal{F} \\ (\nabla u_+ + \mathrm{e}) \cdot \nu = 0 & \text{on } \Gamma_N \times]0,T[\cap \partial Q^+ \\ (\nabla u_+ + \mathrm{e}) \cdot \nu \leq 0 & \text{on } \Gamma_D \times]0,T[\cap \partial Q^+ \cap \{u_+ = 0\} \\ \chi_- \mathrm{e} \cdot \nu = 0 & \text{on } \Gamma_N \times]0,T[\cap \partial Q^- \\ \chi_- = 0 & \text{on } \Gamma_D \times]0,T[\cap \partial Q^- \cap \{\nu \cdot \mathrm{e} > 0\}. \ \square \end{array}$$

4. FINITE SPEED OF PROPAGATION

In this section we give sufficient conditions in order that the support of a solution u grows with finite speed. Roughly speaking, we answer to the following question: knowing that $u(x, t_0) = 0$ near a point x_0, can we deduce that $u(x,t) = 0$ for $t_0 < t < t_0 + \delta$ for some *waiting time* $\delta > 0$ in a smaller neighbourhood ω of x_0? Since u is not continuous in time the question has to be stated in a different way, assuming for instance $\chi < 1$ instead of $u = 0$.

The answer to this question depends on the data and on the location of x_0, which can belong either to Ω, or to Γ_D, or to Γ_N.

In particular the estimates on ω and δ will depend also on the distance of x_0 from some "bad points" which will be studied in the next section. Moreover the optimality of our results will be discussed later on.

Finally the estimates will depend on the norm of the solution u in L^∞. However in [15] it is proved that this norm can be estimated in terms of the data and the geometry of Ω, provided Ω is bounded.

Here and later (u, χ) denotes a solution which is assumed to be bounded, and without loss of generality we take $t_0 = 0$. $\square$

The first case is $x_0 \in \Omega$. This has been considered in [15]:

Theorem 4.1. *Assume $B_R(x_0) \subseteq \Omega$, $u \leq M$ in $\Omega \times]0,T[$, and $0 \leq \chi_0 \leq c$ with $c \in [0,1[$.*
Then there exist $t^* \in]0,T]$ and a continuous function $\rho : [0, t^*[\to]0, R]$, with $\rho(0) = R$, such that

(4.1) $u = 0$ *and* $\partial_t \chi - \operatorname{div}(\chi \mathrm{e}) = 0$

in the subset of $\Omega \times \,]0,t^*[$ *described by the inequality*

(4.2) $|x - x_0| < \rho(t)$. □

Remark 4.2. In fact in [15] it is proved, assuming $|e| = 1$, that $\chi \le c$ in the set (4.2), and this implies (4.1) immediately, taking (1.3) into account. The function ρ is the solution of the following singular Cauchy problem

$$(4.3)\qquad \begin{cases} \rho'(t) = -1 - \dfrac{M}{1-c}\rho(t)^{1-n}\left(\displaystyle\int_{\rho(t)}^{R} \tau^{1-n}\,d\tau\right)^{-1} \\ \rho(0+) = R \end{cases}$$

Existence and uniqueness of a solution ρ defined at least in some interval $]0,t^*[$ are proved in [15].

Clearly t^* and ρ depend only on n, M, c, R, and on an upper bound for $|e|$. Moreover in [15] the following asymptotic behaviour of ρ is also proved:

$$(4.4)\qquad R - \rho(t) = \sqrt{\frac{2M}{1-c}\,t}\,\left(1 + o(1)\right) \qquad \text{as } t \to 0+ .$$

Notice that the support of $u|_{B_R(x_0)\times]0,t^*[}$ is contained in the set described by

$$\rho(t) < |x - x_0| < R, \quad 0 < t < t^*,$$

so that $R - \rho(t)$ estimates from above the maximum width of the part of the support of u contained in the ball $B_R(x_0)$ and (4.4) implies that this width has $t^{1/2}$ as maximum order for small t. □

Now we use the same method to deal with boundary points.

Theorem 4.3. *Assume*

$$x_0 \in \Gamma_D, \qquad B_R(x_0) \cap \Gamma_N = \emptyset$$
$$g = 0 \;\; in \;\; (B_R(x_0) \cap \Gamma_D) \times \,]0,T[, \qquad \chi_0 \le c \;\; in \;\; B_R(x_0) \cap \Omega,$$

where $0 \le c < 1$.
Then the conclusion of Theorem 4.1 *still holds.* □

Proof. Define the functions $\tilde{u}$ and $\tilde{\chi}$ as follows

$$\tilde{u} = \begin{cases} u & \text{in } \Omega \times \,]0,T[\\ 0 & \text{in } \left(\tilde{\Omega} \setminus \Omega\right) \times \,]0,T[\end{cases} \qquad \tilde{\chi} = \begin{cases} \chi & \text{in } \Omega \times \,]0,T[\\ 0 & \text{in } \left(\tilde{\Omega} \setminus \Omega\right) \times \,]0,T[\end{cases}$$

where $\tilde{\Omega} = \Omega \cup B_R(x_0)$, still assuming Γ_N to be the impervious part of $\partial\tilde{\Omega}$.

Take any test function v on $\tilde{\Omega} \times \,]0,T[$ as in the definition of subsolution: then its restriction to $\Omega \times \,]0,T[$ is still a test function according to the definition of solution in $\Omega\times\,]0,T[$. Hence (2.14) holds for $(\tilde{u},\tilde{\chi})$ in $\tilde{\Omega}\times\,]0,T[$. Moreover one easily checks (2.9) using $e\cdot\nu\geq 0$. Finally all the other conditions required in the definition of subsolution are clearly satisfied with obvious new data. Hence $(\tilde{u},\tilde{\chi})$ is a subsolution in $\tilde{\Omega}\times\,]0,T[$.

Using the same supersolution constructed in [15] and applying the comparison theorem 2.4 we get the result. □

In order to deal with points of the impervious boundary we assume that

(4.5) $\overline{\Gamma}_N$ is a C^1 submanifold of Γ.

Lemma 4.4. *There exists $R_1>0$, depending only on the geometry of $\overline{\Gamma}_N$, such that for any $R\in\,]0,R_1/2[$ and any $x,x_0\in\overline{\Gamma}_N$ satisfying*

$$(4.6)\qquad \left|x-\left(x_0-\frac{R}{2}\nu(x_0)\right)\right|<R$$

we have

$$(4.7)\qquad \frac{x-\left(x_0-\frac{R}{2}\nu(x_0)\right)}{\left|x-\left(x_0-\frac{R}{2}\nu(x_0)\right)\right|}\cdot\nu(x)\geq\frac{1}{16}.\ \square$$

Proof. Take R_1 such that for $x,x_0\in\overline{\Gamma}_N$ satisfying $|x-x_0|<R_1$. Then

$$\left|\frac{x-x_0}{|x-x_0|}\cdot\nu(x_0)\right|\leq\frac{1}{8},\quad |\nu(x)-\nu(x_0)|\leq\frac{1}{8},\quad \nu(x)\cdot\nu(x_0)\geq\frac{7}{8}.$$

Choose R, x, and x_0 as in the statement and set $x_1=x_0-\frac{R}{2}\nu(x_0)$. Then $|x-x_0|\leq\frac{3}{2}R<R_1$, so that

$$(x-x_1)\cdot\nu(x)=$$

$$|x-x_0|\frac{x-x_0}{|x-x_0|}\cdot\nu(x_0)+(\nu(x)-\nu(x_0))+\frac{R}{2}\nu(x_0)\cdot\nu(x)\geq$$

$$\geq-\frac{1}{8}\cdot\frac{3}{2}R-\frac{1}{8}\cdot\frac{3}{2}R+\frac{7}{8}\cdot\frac{R}{2}=\frac{R}{16}.$$

Hence (4.7) follows. □

Theorem 4.5. *Assume*

$$x_0\in\overline{\Gamma}_N,\quad g=0\quad in\quad (B_R(x_0)\cap\Gamma_D)\times\,]0,T[,\quad and$$

$$\chi_0=0\quad in\quad B_R(x_0)\cap\Omega.$$

Then there exist $t^*\in\,]0,T]$ and a continuous function $\rho:[0,t^*[\;\to\;]0,R]$ such that

(4.8) $u = 0$ *and* $\chi = 0$

in the subset of $\Omega \times]0, t^*[$ *described by the inequality* (4.2). *Moreover* t^* *and* ρ *depend only on* n, R, T, $|e|$, *and on an upper bound for* u *in* $\Omega \times]0, T[$. □

Proof. Fix R_1 as in the lemma and $r_0 < \frac{1}{2}\min\{R_1, R\}$, and consider any $M \geq \sup u$ (which will be chosen later). We construct a supersolution in $B_{r_0}(x_1)$, where

$$x_1 = x_0 - \frac{r}{2}\nu(x_0).$$

To do this we solve the singular Cauchy problem (4.3) (assuming $|e| \leq 1$) with r_0 in place of R: we find a solution, which we denote by r, in some interval $]0, t_0[\subseteq]0, T[$.

Following [15], for any $t \in]0, t_0[$ we define $\overline{u}$ in $B_{r_0}(x_1)$ by means of

$$\overline{u}(t) \in C^0\left(\overline{B_{r_0}(x_1)}\right), \quad \Delta\overline{u} = 0 \text{ in } B_{r_0}(x_1) \setminus \overline{B_{r(t)}(x_1)}$$

$$\overline{u}(t) = 0 \text{ in } B_{r(t)}(x_1), \quad \text{and} \quad \overline{u} = M \text{ on } \partial B_{r_0}(x_1),$$

and take as $\overline{\chi}$ the characteristic function of the set where $\overline{u}(t) > 0$.

We claim that $(\overline{u}, \overline{\chi})$ is a supersolution in $(B_{r_0}(x_1) \cap \Omega) \times]0, t^*[$, taking $\Gamma_N \cap B_{r_0}(x_1)$ as impervious boundary and choosing a smaller t^* (if necessary) satisfying

(4.9) $t^* \leq T$ and $r(t) \leq \dfrac{r_0}{2}$ for $0 < t < t^*$.

To prove this we use 3.1–3.3 with

$$F(x, t) = |x - x_1| - r^2(t).$$

From (4.3), we have for $|x - x_1| = r(t)$

$$r'(t) + \frac{x - x_1}{|x - x_1|} \cdot e + M \frac{r(t)^{1-n}}{\displaystyle\int_{r(t)}^{r_0} \tau^{1-n}\, d\tau} \leq 0.$$

But in $\{\overline{u} > 0\}$ we have

$$\nabla\overline{u}(x, t) = M \frac{|x - x_1|^{1-n}}{\displaystyle\int_{r(t)}^{r_0} \tau^{1-n}\, d\tau} \frac{x - x_1}{|x - x_1|}$$

so that

$$\partial_t F - \nabla F \cdot e - \nabla\overline{u} \cdot \nabla F \geq 0 \quad \text{for } |x - x_1| = r(t).$$

In particular

$$\partial_t F - \nabla F \cdot \mathrm{e} \geq \nabla \overline{u} \cdot \nabla F \geq 0 \quad \text{for } |x - x_1| = r(t),$$

which allows us to apply Proposition 3.1.

Now we prove that $(\nabla \overline{u} + \mathrm{e}) \cdot \nu \geq 0$ at points (x,t) with $x \in \Gamma_N$ and $r(t) < |x - x_1| < r_0$, provided M is suitably chosen. Thanks to the lemma we have

$$(\nabla \overline{u} + \mathrm{e}) \cdot \nu = \left(M \frac{|x - x_1|^{1-n}}{\int_{r(t)}^{r_0} \tau^{1-n}\, d\tau} \frac{x - x_1}{|x - x_1|} + \mathrm{e} \right) \geq$$

$$\geq \frac{M}{16} \frac{|x - x_1|^{1-n}}{\int_{r(t)}^{r_0} \tau^{1-n}\, d\tau} \frac{x - x_1}{|x - x_1|} - 1 \geq \frac{M}{16} \frac{r_0^{1-n}}{\int_{r_0/2}^{r_0} \tau^{1-n}\, d\tau} \frac{x - x_1}{|x - x_1|} - 1,$$

which is nonnegative with an obvious choice of M.

Hence 3.1–3.3 and the comparison theorem 2.4 give $u \leq \overline{u}$ and $\chi \leq \overline{\chi}$ in $(B_{r_0}(x_1) \cap \Omega) \times \,]0, t^*[$, and the result follows taking $\rho(t) = r(t) - r_0/2$. □

Remark 4.6. As the singular Cauchy problem has not been changed we have the same asymptotic estimates on the waiting time t^* as for points in Ω and in Γ_D. □

5. CASES OF NO WAITING TIME

The first case deals with a point $x_0 \in \Gamma_D$ where the boundary datum g is positive. The following result, which seems obvious, has actually to be proved, since non continuity properties of u are known. We need here the assumption

(5.1) $\quad \Gamma_D$ is a $C^{1,\alpha}$ submanifold of Γ

Theorem 5.1. *Assume*

$$x_0 \in \Gamma_D, \qquad B_R(x_0) \cap \Gamma_N = \emptyset,$$

$$g(x,t) \geq M(x,t) > 0 \quad \text{in } B_R(x_0) \cap \Gamma_D.$$

Assume moreover either $\nu(x_0) \cdot \mathrm{e} > 0$ or $\mathrm{e} = 0$.
Then there exist $t^* \in \,]0, T]$ and a continuous function $\rho : [0, t^*[\, \to \,]0, R]$ such that

(5.2) $\quad u(x,t) > 0 \quad$ *for* $|x - x_0| < \rho(t)$ *and* $0 < t < t^*$.

Moreover t^* and ρ depend only on n, R, T, M, and the geometry. □

Proof. Let us assume first $e \neq 0$: thus $\nu(x_0) \cdot e > 0$. We can represent

$$x = (x', y) \quad \text{with} \quad x' \in \mathbb{R}^{n-1} \quad \text{and} \quad y = x \cdot \frac{e}{|e|}$$

and assume that Γ_D has the form $y = f_0(x')$ in the cylinder

$$C(x_0) = \{x : \ |x' - x_0'| < \delta, \ |y - y_0| < 3\varepsilon\},$$

where (x_0', y_0) is the point x_0. We can also assume

$$C(x_0) \subset B_R(x_0) \quad \text{and} \quad |f_0(x') - y_0'| \leq \varepsilon \quad \text{for} \quad |x' - x_0'| < \delta.$$

Define

$$f(x') = f_0(x') + \varepsilon\delta^{-2}|x' - x_0'|^2 \quad \text{and} \quad C^+(x_0) = C(x_0) \cap \{y > f(x')\},$$

and solve the Dirichlet problem

$$\Delta w = 0 \quad \text{in} \quad C^+(x_0) \qquad \text{and} \qquad w = w_0 \quad \text{on} \quad \partial C^+(x_0),$$

where a smooth $w_0 \not\equiv 0$ is fixed satisfying

$$0 \leq w_0 \leq 1 \quad \text{on} \quad \partial C^+(x_0) \quad \text{and} \quad w_0 = 0 \quad \text{on} \quad \partial C^+(x_0) \cap \{y = f(x')\}.$$

Setting

$$K = \{(x', f(x')) : \ |x' - x_0'| \leq \delta/2\},$$

w is smooth up to K and satisfies the Hopf maximum principle on K. Hence, setting

$$\alpha = \inf_K |\partial_\nu w|,$$

we have $\alpha > 0$. Define then the function

$$F(x,t) = F(x', y, t) = y - f(x') + |e|t + \alpha \int_0^t M(\tau)\, d\tau,$$

the set

$$\omega = \{(x', y) \in \Omega : \ |x' - x_0'| < \delta, \ |y' - y_0'| < 2\varepsilon\},$$

and the function $\underline{u}$ on $\omega \times \,]0, t^*[$

$$\underline{u}(x,t) = \begin{cases} M(t) \cdot w\left(x', y + |e|t_0 + \alpha \int_0^t M(\tau)\, d\tau\right) & \text{if } F(x,t) > 0 \\ 0 & \text{if } F(x,t) < 0, \end{cases}$$

where $t^* \leq T$ is such that

$$|e|t^* + \alpha \int_0^{t^*} M(\tau)\,d\tau \leq \varepsilon/4.$$

Calling $\underline{\chi}$ the characteristic function of the set $\{\underline{u} > 0\}$, we claim that $(\underline{u}, \underline{\chi})$ is a subsolution in $\omega \times]0, t^*[$, assuming the whole of $\partial\omega$ to be pervious.

It is easy to see that the conditions of Proposition 3.1 hold. Moreover on $\{F = 0\}$ we have

$$\partial_t F - \nabla F \cdot e - \nabla \underline{u} \cdot \nabla F \leq 0,$$

since ∇F, $\nabla \underline{u}$, and ν have the same direction and $|\partial_\nu u| \geq \alpha M(t)$.

Finally we note that $w \leq 1$ by the maximum principle; hence $\underline{u}(x,t) \leq M(t) \leq g(x,t)$ on $\Gamma_D \cap \partial\omega$. On the other hand $\underline{u} = 0$ elsewhere on $\partial\omega$: therefore $\underline{u} \leq u$ on $\partial\omega$.

We can thus apply 3.1–3.3 and the comparison theorem 2.4, and the result follows with an obvious ρ.

If $e = 0$ the same proof works, choosing the coordinate system e.g. such that $y = x \cdot \nu(x_0)$. □

Remark 5.2. Going back to the previous proof we observe that the positivity of α plays a crucial role only if $e = 0$. Therefore in the opposite case the same proof still works taking $\alpha = 0$, so that the smoothness of Γ_D near x_0 is essentially not needed. □

From the previous proof we can see that, in the case $\nu(x_0) \cdot e > 0$, $\rho(t)$ behaves essentially like $|e|t + \alpha \int_0^t M(\tau)\,d\tau$ as $t \to 0+$. This estimate is not optimal, at least when $\inf M > 0$: in this case one expects indeed $\rho(t) \sim t^{1/2}$. This improvement is shown in the following theorem.

Theorem 5.3. *Assume*

$$x_0 \in \Gamma_D, \qquad x_1 \in \mathbb{R}^n \setminus \overline{\Omega}, \qquad r_0 = |x_1 - x_0|,$$

$$B_{r_0}(x_1) \cap \Omega = \emptyset, \qquad R > r_0, \qquad B_R(x_1) \cap \Gamma_N = \emptyset,$$

$$g(x,t) \geq M > 0 \qquad \text{on} \quad (B_R(x_0) \cap \Gamma_D) \times]0, T[.$$

Then there exist $t^* \in]0, T]$ *and a continuous function* $\rho :]0, t^*[\to \mathbb{R}^+$ *such that* $\rho(0+) = 0$ *and*

$$u(x,t) > 0 \quad \text{for} \ |x - x_0| < \rho(t) \ \text{and} \ 0 < t < t^*.$$

Moreover t^* *and* ρ *depend only on* n, R, r_0, T, M, $|e|$, *and we have*

$$\rho(t) = \sqrt{2Mt}\left(1 + o(1)\right) \quad \text{as} \ t \to 0+. \qquad (5.3)$$
□

Proof. Consider the singular Cauchy problem

$$r'(t) = -|e| + \frac{M\, r(t)^{1-n}}{\displaystyle\int_0^{r(t)} \tau^{1-n}\, d\tau}, \qquad r(0+) = r_0.$$

It is easy to see that a unique solution (with $r(t) > r_0$ for $t > 0$) exists in some interval $]0, t_0[$ (see the previous sections).

Choose $t^* \leq \min\{t_0, T\}$ such that

$$r(t^*) \leq R \qquad \text{and} \qquad r'(t) \geq |e| \quad \text{for } 0 < t < t^*.$$

For $0 < t < t^*$ define $\underline{u} \in C^0\left(\overline{\Omega \cup B_R(x_1)}\right)$ by

$$\begin{cases} \underline{u}(t) = M & \text{in } B_{r_0}(x_1) \\ \Delta \underline{u}(t) = 0 & \text{in } B_{r(t)}(x_1) \setminus B_{r_0}(x_1) \\ \underline{u}(t) = 0 & \text{in } \Omega \setminus B_{r(t)}(x_1) \end{cases}$$

and call $\underline{\chi}(t)$ the characteristic function of $B_{r(t)}(x_1)$.

We claim that $(\underline{u}, \underline{\chi})$ is a subsolution on $\Omega \times]0, t^*[$. Setting

$$F(x,t) = r^2(t) - |x - x_1|^2,$$

we have

$$\partial_t F - \nabla F \cdot e \geq 2r(r' - |e|) \geq 0 \quad \text{where} \quad F = 0.$$

Moreover on the same set we have

$$\partial_t F - \nabla F \cdot e - \nabla \underline{u} \cdot \nabla F \leq$$

$$2rr' + 2r|e| - 2(x - x_1) \cdot \frac{M|x - x_1|^{1-n}}{\displaystyle\int_0^{r} \tau^{1-n}\, d\tau} \frac{x - x_1}{|x - x_1|} \leq 0.$$

As $\underline{u} \leq u$ on Γ_D, $(\underline{u}, \underline{\chi})$ is a subsolution by 3.1–3.3 and 2.4 gives $\underline{u} \leq u$. Hence we can take $\rho(t) = r(t) - r_0$ and the estimate (5.3) is obtained as in the previous sections. □

Now we deal with points of Γ_N and assume that χ_0 is strictly positive there. We deduce that u becomes strictly positive under the crucial assumption $\nu \cdot e < 0$, which excludes the case $e = 0$ and means that Γ_N is not vertical for the dam problem.

Theorem 5.4. *Assume*

$$x_0 \in \Gamma_N, \qquad B_R(x_0) \cap \Gamma_D = \emptyset, \qquad \nu(x_0) \cdot e < 0,$$

$$r_1 > 0, \qquad B_{r_1}(x_0 + r_1 \nu(x_0)) \cap \Omega = \emptyset,$$

$$\chi_0 \geq c \quad \text{in } B_R(x_0) \cap \Omega$$

with $0 < c < 1$.

Then there exist $t^* \in \,]0,T]$ *and a continuous function* $\rho : \,]0,t^*[\, \to \mathbb{R}^+$ *such that* $\rho(0+) = 0$ *and*

(5.4) $u(x,t) > 0$ *for* $|x - x_0| < \rho(t)$ *and* $0 < t < t^*$.

Moreover t^* *and* ρ *depend only on* n, R, T, $\nu(x_0)$, $|\mathrm{e}|$, *and the geometry.* □

Proof. We may assume $|\mathrm{e}| = 1$ and $B_{r_1}(x_0 + r_1\nu(x_0)) \subset B_R(x_0)$. We set

$$r_0 = r_1/2, \qquad x_1 = x_0 + r_0\nu(x_0), \qquad B' = B_{r_0}(x_1)$$

$$B'' = B_{r_1}(x_0 + r_1\nu(x_0)), \qquad \text{and} \qquad B''' = B_{3r_0}(x_1),$$

and observe that $B''' \subset B_R(x_0)$.

Now we fix $\varepsilon > 0$ with $(1-\varepsilon)^2 > 1 - c$, we consider the set described by the inequalities

$$0 < \frac{x - x_1}{|x - x_1|} \cdot \mathrm{e} < \frac{|\nu(x_0) \cdot \mathrm{e}|}{1 - \varepsilon},$$

we denote by A its connected component containing x_0, and set

$$r^* = \inf_{x \in \partial B'' \cap \partial A} |x - x_1|.$$

Hence $r^* < 3r_0$ and, for $0 < r < r^*$, we have

$$\Gamma_N \cap B_r(x_1) \subseteq \overline{B_r(x_1)} \setminus B'' \subset A$$

$$\Omega \cap \partial B_r(x_1) \subseteq \overline{B_r(x_1)} \setminus B'' \subset A.$$

Consider the Cauchy problem

$$\text{(5.5)} \qquad \begin{cases} r'(t) = -\dfrac{|\nu(x_0)\cdot \mathrm{e}|}{1-\varepsilon} + \dfrac{(1-\varepsilon)|\nu(x_0)\cdot \mathrm{e}|}{1-c} \, \dfrac{(r(t) - r_0) r(t)^{1-n}}{\displaystyle\int_{r_0}^{r(t)} \tau^{1-n}\, d\tau} \\[2ex] r(0) = r_0. \end{cases}$$

As the right hand side is not singular as $r(t) = r_0$, a unique solution exists on some interval $]0, t_0[$. We have

$$r'(0) = |\nu(x_0)\cdot \mathrm{e}| \left(-\frac{1}{1-\varepsilon} + \frac{1-\varepsilon}{1-c} \right) > 0,$$

so that $t_1 \in \,]0, t_0]$ exists such that

$$r'(t) > 0 \quad \text{and} \quad r(t) < r^* \qquad \text{for } 0 < t < t_1.$$

For $0 < t < t_1$ we define $\underline{u}(t) \in C^0(\mathbb{R}^n)$ by

$$\begin{aligned}
&\underline{u}(t) = (1-\varepsilon)|\nu(x_0)\cdot \mathrm{e}|(r(t)-r_0) && \text{for } |x-x_1|<r_0\\
&\Delta\underline{u}(t) = 0 && \text{for } r_0<|x-x_1|<r(t)\\
&\underline{u}(t) = 0 && \text{for } |x-x_1|>r(t),
\end{aligned}$$

and set $\Omega' = \Omega \cap A \cap B'''$.

Finally let $\underline{\chi_0}$ be the product of the characteristic function of Ω' by ε and set for $x \in \Omega \cup B_{r_0}(x_1)$ and $t > 0$

$$\underline{\chi}(x,t) = \begin{cases} \underline{\chi_0}(x+t\mathrm{e}) & \text{if } \underline{u}(x,t)=0 \\ 1 & \text{if } \underline{u}(x,t)>0. \end{cases}$$

Noting that the subsets of $\Omega \cup B_{r_0}(x_1)$ where $\underline{\chi}(\cdot,t)=0$ and respectively $\underline{\chi}(\cdot,t)=1$ have a positive distance at the time $t=0$ and that they move continuously, the same is true for $0<t<t_2$ with some $t_2 \in \,]0,t_0[$.

Moreover, as Γ_N lies outside B'' (whose radius is $>r_0$), the diameter of $\Gamma_N \cap B_{r(t)}(x_1)$ tends to 0 as $t \to 0$, whence there exists $t_3>0$ such that

$$|\nu(x)\cdot\mathrm{e}| \geq (1-\varepsilon)^{1/2}|\nu(x_0)\cdot\mathrm{e}| \quad \text{on } \Gamma_N \cap B_{r(t)}(x_1) \quad \text{for } 0<t<t_3$$

and such that

$$\frac{r_0^{1-n}}{r(t)^{1-n}} \leq (1-\varepsilon)^{-1/2} \qquad \text{for } 0<t<t_3.$$

Hence we chose $t^* = \min\{t_1,t_2,t_3\}$.

We claim that the restriction of $(\underline{u},\underline{\chi})$ to $\Omega' \times \,]0,t^*[$ is a subsolution assuming the subset of $\partial\Omega'$ where $\nu\cdot\mathrm{e}<0$ as impervious part of the boundary.

Setting $F(x,t) = r^2(t) - |x-x_1|^2$, we have on $\{F=0\}$

$$\partial_t F - \nabla F\cdot\mathrm{e} = 2rr' + 2(x-x_1)\cdot\mathrm{e} \geq 0 \qquad \text{and}$$

$$(1-c)(\partial_t F - \nabla F\cdot\mathrm{e}) - \nabla\underline{u}\cdot\nabla F =$$

$$= (1-c)\left[2rr' + 2(x-x_1)\cdot\mathrm{e}\right] -$$

$$-2(x-x_1)\cdot\frac{x-x_1}{|x-x_1|}\,\frac{(1-\varepsilon)|\nu(x_0)\cdot\mathrm{e}|(r-r_0)r^{1-n}}{\displaystyle\int_{r_0}^{r}\tau^{1-n}\,d\tau} \leq$$

$$(1-c)2rr' + \frac{1-c}{1-\varepsilon}\,2r|\nu(x_0)\cdot\mathrm{e}| - \frac{2(1-\varepsilon)|\nu(x_0)\cdot\mathrm{e}|(r-r_0)r^{2-n}}{\displaystyle\int_{r_0}^{r}\tau^{1-n}\,d\tau} = 0$$

thanks to (5.5).

On the bottom of Ω' we have either $\nabla\underline{u}=0$ and $\underline{\chi}\mathrm{e}\cdot\nu \leq 0$ or $\underline{u}>0$ and

$$(\nabla \underline{u} + \mathrm{e}) \cdot \nu =$$

$$= \left(- \frac{(1-\varepsilon)|\nu(x_0)\cdot \mathrm{e}|(r-r_0)|x-x_1|^{1-n}}{\displaystyle\int_{r_0}^{r} \tau^{1-n}\, d\tau} \frac{x-x_1}{|x-x_1|} + \mathrm{e} \right) \cdot \nu \le$$

$$\le (1-\varepsilon)|\nu(x_0)\cdot \mathrm{e}| \frac{r_0^{1-n}}{r^{1-n}} \frac{x-x_1}{|x-x_1|} \cdot \nu(x) + \nu(x)\cdot \mathrm{e} \le$$

$$\le (1-\varepsilon)|\nu(x)\cdot \mathrm{e}|(1-\varepsilon)^{-1/2}\cdot(1-\varepsilon)^{-1/2} + \nu(x)\cdot \mathrm{e} = 0.$$

On the pervious part of $\partial\Omega'$ we have $\nabla \underline{u} = 0$ and either $\nu\cdot \mathrm{e} = 0$ or $\underline{\chi} = 0$, whence $(\nabla \underline{u} + \underline{\chi}\mathrm{e})\cdot \nu = 0$.

Therefore 3.1–3.3 and 2.4 give $u \ge \underline{u}$ in $\Omega' \times \,]0, t^*[$ and we can take $\rho = r - r_0$. □

In the case $c = 1$ we have a stronger result, which however can be proved simply modifying a little bit the previous argument as follows: we fix $\varepsilon \in \,]0,1[$ and define $r \in \,]0, r^*[$, independent of t, just in order that

$$\left(- \frac{(1-\varepsilon)|\nu(x_0)\cdot \mathrm{e}|(r-r_0)|x-x_1|^{1-n}}{\displaystyle\int_{r_0}^{r} \tau^{1-n}\, d\tau} \frac{x-x_1}{|x-x_1|} + \mathrm{e} \right) \cdot \nu \le 0 \quad \text{on } \Gamma_N \cap B_r(x_1).$$

In particular $\underline{u}$ will not depend on t.

This leads to the following result

Theorem 5.5. *Under the assumptions of Theorem 5.4 with $c = 1$, there exist three strictly positive constants ρ, t^*, and m such that*

$$u \ge m \qquad \text{in} \quad (\Omega \cap B_\rho(x_0)) \times \,]0, t^*[.$$

These constants depend only on R, $\nu(x_0)$, $|\mathrm{e}|$, and the geometry. □

Remark 5.6. Theorems 5.4 and 5.5 give more general conditions in order that the solution behaves like in the examples of [15] (Prop. 3.8 and Rem. 3.9). In the last case u was discontinuous in time. This negative fact arises in general whenever the conditions of Theorem 5.5 are satisfied at some $t = t_0$ and $u = 0$ for $t < t_0$. □

Remark 5.7. The result given by Theorem 5.5 can be improved.

Assume $t_0 > 0$ and $\chi_0(\cdot, t_0) = 1$ in a connected open set Ω_0 whose boundary contains $\Gamma_N \cap B_{2R}(x_0)$. Because of (2.9) we have $\chi(\cdot, t) = 1$ in an open set Ω_t which is close to Ω_0 for $t_0 < t < t_0 + \delta$ for some $\delta > 0$, and thus still connected. Hence $\chi = 1$ and $\Delta u = 0$ in the union $\mathcal{U}$ over $]t_0, t_0 + \delta[$ of $\Omega_t \times \{t\}$.

If t^*, ρ, and m are given by 5.5, we have $u \ge w$ in $\mathcal{U}$, where $w(t)$ is the harmonic function in Ω_t such that $w(t) = m$ on $\Gamma_N \cap B_\rho(x_0)$ and $w(t) = 0$

elsewhere on $\partial\Omega_t$. In particular u is discontinuous in time at *every* $x \in \Omega_0$ for $t = t_0$.

Notice that a similar situation happens when a moving subdomain Ω_t where $\chi = 1$ and $u = 0$ meets a region where $u > 0$. □

6. GLOBAL RESULTS

Collecting the results of section 4 we can state the following

Theorem 6.1. *Fix $\varepsilon \in \,]0,1[$ and $T > 0$, and define G to be the projection on $\mathbb{R}^n$ of the support of the restriction of g to $\Gamma_D \times \,]0,T[$. Define moreover*

$$E_1 = \overline{G} \cup \overline{\operatorname{supp}(\chi_0 - (1-\varepsilon))^+} \cup \overline{\Gamma_N}$$

$$E_2 = \overline{G} \cup \overline{\operatorname{supp}\chi_0}$$

and let E be either E_1 or E_2.
Then there exist $t^* \in \,]0,T[$ and $c > 0$, depending only on T, ε, the geometry, and upper bounds for u on $\Omega \times \,]0,T[$ and for $|\mathrm{e}|$, such that for any $x \in \overline{\Omega} \setminus E$ and $t \in \,]0,t^*[$ with $d(x,E) \geq c\sqrt{t}$ we have

$$\chi(\cdot,\tau) < 1 \qquad in \;\; B_{d(x,E)-c\sqrt{t}}(x) \quad \forall \tau \in \,]0,t[. \;\square$$

Proof. If $x \in \Omega$ apply Theorem 4.1 and Remark 4.2; if $x \in \Gamma_D$ apply Theorem 4.3; if $x \in \Gamma_N$ apply Theorem 4.5 and Remark 4.6. □

Remark 6.2. This result is optimal since it is easy to see the following (with simple changes in the examples of [26]). Take

$$\Omega = \Omega' \times \,]-h,0[, \qquad \Gamma_N = \partial\Omega' \times \,]-h,0[, \qquad \mathrm{e} = (0,1),$$

$$g \in C^0[0,\infty[\;\; \text{strictly positive,} \quad \text{and} \quad \chi_0 \equiv c \;\; \text{with } 0 \leq c < 1,$$

where $\Omega' \in \mathbb{R}^{n-1}$. Then the solution is given by

$$u(x,t) = \frac{g(t)}{\varphi(t)}\,(x \cdot \mathrm{e} + \varphi(t))^+$$

$$\chi(x,t) = \begin{cases} 1 & \text{where } \; x \cdot \mathrm{e} > -\varphi(t) \\ c & \text{where } \; x \cdot \mathrm{e} < -\varphi(t) \end{cases}$$

where φ solves the singular Cauchy problem

$$\varphi'(t) = 1 + \frac{g(t)}{(1-c)\varphi(t)}, \qquad \varphi(0+) = 0.$$

This holds until $\varphi(t) < h$. One sees that

$$\varphi(t) = \sqrt{\frac{2g(0)}{1-c}\,t}\,\Big(1 + o(1)\Big) \quad \text{as } t \to 0,$$

and this shows the optimality of our result and in particular that the dependence on ε cannot be avoided. □

One could wonder whether or not it is sufficient in Theorem 6.1 to have x far from the set $\{\chi_0 = 1\}$ instead of $\{\chi_0 \geq 1 - \varepsilon\}$: we present now an example allowing a waiting time without the correct asymptotic estimate.

Example 6.3. Consider the above example with $g \equiv 1$ and replace χ_0 by $\chi_0(x) = \psi(x \cdot \mathrm{e})$ with a continuous ψ satisfying $0 < \psi < 1$ in $]-h, 0[$. Then the solution is given by similar formulae with φ such that

$$\varphi'(t) = 1 + \frac{1}{\big(1 - \psi(t - \varphi(t))\big)\varphi(t)}, \qquad \varphi(0+) = 0.$$

Given any $\alpha \in \,]0, 1/2[$ we choose properly h and ψ in order that $\varphi(t) = t^\alpha$ for small t. Simple computations show that the proper condition is

$$\psi(t - t^\alpha) = 1 - t^{1-2\alpha};$$

hence we take

$$t_0 = \min_{0<t<1} \{t - t^\alpha\}, \quad h = |t_0 - t_0^\alpha|, \quad \text{and} \quad \psi(s) = 1 - \vartheta^{1-2\alpha}(s),$$

where ϑ is the inverse function of $t \mapsto t - t^\alpha$, $0 < t < t_0$. □

Now we state a property of the *mushy region* (even though this word is more appropriate for other problems, see [21])

$$M(t) = \{x \in \Omega : \ \exists r > 0 : \ B_r(x) \subset \Omega, \ 0 < \chi(\cdot, t) < 1 \text{ in } B_r(x)\}.$$

Theorem 6.4. *The function $t \mapsto M(t) + t\mathrm{e}$ is nonincreasing in the sense of inclusion. In particular*

$$M(t) + t\mathrm{e} \subseteq M(0) \subseteq \Omega \qquad \forall t > 0.$$

Moreover if $t > 0$ and $M(t) \neq \emptyset$ we have

$$\partial_t \chi - \operatorname{div}(\chi \mathrm{e}) = 0 \quad \text{in} \quad \{(x, \tau) \in \Omega \times \,]0, t[\, : \ x \in M(\tau)\}.$$

Finally, if $\mathrm{e} \neq 0$, we have

$$M(t) = \emptyset \quad \forall t \geq \frac{1}{|\mathrm{e}|} \operatorname*{osc}_{x \in \Omega} x \cdot \mathrm{e}. \ \square$$

Proof. Let $t \geq 0$, $h > 0$, and $B_r(x) \subseteq M(t + h)$. Consider the cylinder

$$C = \{(\xi,\tau) \in \mathbb{R}^n \times]t, t+h[\; : \; \xi - \tau e \in B_r(x)\}.$$

Using (2.9) and taking into account the sign of $\nu \cdot e$, we deduce that $\chi < 1$ in $C \cap (\Omega \times]t, t+h[)$. Thus $u = 0$ there, whence $\partial_t \chi - \operatorname{div}(\chi e) = 0$ and thus $0 < \chi < 1$ in the same set.

Therefore C is contained in $\Omega \times]t, t+h[$, otherwise the intersection of C and $(\Gamma_D \cap \{e \cdot \nu > 0\}) \times]t, t+h[$ would have non empty interior and the overflow condition would not be satisfied (note that in this case the overflow condition has the form $\chi e \cdot \nu \leq 0$). In particular $B_r(x + he) \subseteq M(t)$. This shows that $M(t) + te \supseteq M(t+h) + (t+h)e$.

We can go directly to the last point of the statement. If $t|e| \geq \operatorname{osc}_{x \in \Omega} x \cdot e$ (which is the width of Ω in the e-direction) $\Omega + te$ does not intersect Ω. □

Remark 6.5. If $e = 0$ there can be a steady mushy region, taking for instance $u = 0$ and χ independent of t with compact support. □

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