Publicacions Matemàtiques, Vol 40 (1996), 527–560.

# UNIQUENESS AND EXISTENCE OF SOLUTIONS IN THE $BV_t(Q)$ SPACE TO A DOUBLY NONLINEAR PARABOLIC PROBLEM

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*Abstract*

In this paper we present some results on the uniqueness and existence of a class of weak solutions (the so called BV solutions) of the Cauchy-Dirichlet problem associated to the doubly nonlinear diffusion equation

$$b(u)_t - \operatorname{div}(|\nabla u - k(b(u))\boldsymbol{e}|^{p-2}(\nabla u - k(b(u))\boldsymbol{e})) + g(x,u) = f(t,x).$$

This problem arises in the study of some turbulent regimes: flows of incompressible turbulent fluids through porous media, gases flowing in pipelines, etc. The solvability of this problem is established in the $BV_t(Q)$ space. We prove some comparison properties (implying uniqueness) when the set of jumping points of the BV solution has $N$-dimensional null measure and suitable additional conditions as, for instance, $b^{-1}$ locally Lipschitz. The existence of this type of weak solution is based on suitable uniform estimates of the BV norm of an approximated solution.

## 1. Introduction

Let be $\Omega$ a bounded open subset of $\mathbb{R}^N$ with regular boundary and $T > 0$. We consider the following Cauchy-Dirichlet problem

$$b(u)_t - \operatorname{div}\phi(\nabla u - k(b(u))e) + g(x,u) = f(t,x) \text{ in } Q, \tag{1.1}$$

$$u(t,x) = 0 \text{ on } \Sigma, \tag{1.2}$$

$$u(0,x) = u_0(x) \text{ in } \Omega, \tag{1.3}$$

Both authors are partially supported by the DGICYT (Spain) Project No. PB 93/0443.

where $Q := ]0,T[\times\Omega$ and $\Sigma := ]0,T[\times\partial\Omega$. We also assume some structure conditions such as the ellipticity of the diffusion operator (which is implied by the monotonicity of the power vectorial function

$$\phi(\xi) = |\xi|^{p-2}\xi \quad \forall \xi \in \mathbb{R}^N \tag{1.4}$$

with $p > 1$) and the monotonicity of the real continuous function $b$. In fact, we shall assume that

$$b \text{ is continuous strictly increasing and } b(0) = 0. \tag{1.5}$$

The continuous functions $k$ and $g(x,\cdot)$ are assumed satisfying some additional assumptions (see (2.1)-(2.5)). In (1.1) $\boldsymbol{e}$ denotes a given unit vector in $\mathbb{R}^N$.

Problem (1.1), or some special cases of it, arises in many different physical contexts. For instance, the flow of a gas through a porous medium in a turbulent regime is described by the continuity equation

$$\theta_t + \operatorname{div} \boldsymbol{v} = 0$$

and the nonlinear Darcy's law

$$|\boldsymbol{v}|^{q-2}\boldsymbol{v} = -K(\theta)\nabla\Phi(\theta), \text{ for some } q > 2,$$

where $\theta(t,x)$ is the volumetric moisture content, $\boldsymbol{v}$ the velocity, $K(\theta)$ the hydraulic conductivity and $\Phi$ the total potential (usually given by $\Phi(\theta) = \psi(\theta) + z$ with $\psi(\theta)$ the hydrostatic potential and $z$ the gravitational potential; see Volker [**V69**]). If $\boldsymbol{e}$ denotes the unit vector in the vertical direction, we obtain

$$\theta_t - \operatorname{div}(|\nabla\varphi(\theta) - K(\theta)\boldsymbol{e}|^{p-2}(\nabla\varphi(\theta) - K(\theta)\boldsymbol{e})) = 0$$

with

$$\varphi(\theta) = \int_0^\theta K(s)\Phi'(s)\,ds, \text{ and } p = \frac{q}{q-1}$$

(notice that now $1 < p < 2$). Functions $\varphi$ and $k$ are given by physical experiments. Usually they are nondecreasing functions. For unsaturated media $\varphi$ is strictly increasing. In that case, the function $u = \varphi(\theta)$ verifies the equation (1.1) with $b = \varphi^{-1}$, $K = k$ and $g = f = 0$.

Another typical example is given by a turbulent gas flowing in a pipeline. If we assume that the gas is perfect and the pipe has uniform cross sectional area, we arrive to the system

$$\rho_t + (\rho v)_x = 0$$
$$\rho v_t + \rho v v_x + P_x = -\frac{\lambda}{2}\rho|v|v$$

where $\rho$, $P$ and $v$ are the density, pressure, and velocity of the gas which are unknown functions depending on the scalar $x$ (the distance along the pipe) and time $t$. Using asymptotic methods, it was shown in Díaz-Liñán **[DLi89]** that for large values of time the term $\rho v_t + \rho v v_x$ can be neglected and so the second equation may be replaced by $P_x = -\frac{\lambda}{2}\rho|v|v$. Then, if we define $u = |P|P$, $u$ satisfies equation (1.1) with $b(u) = u^{1/2}\operatorname{sign}(u)$, $k = g = f = 0$ and $p = 3/2$. We notice that in this case $b^{-1}$ is locally Lipschitz.

Previous results on the mathematical treatment of problem (1.1), (1.2) and (1.3) can be found in the references of the paper Díaz-de Thelin **[DT94]** where the authors pay an special attention to the stabilization question. The main goal of this paper is to improve the uniqueness results of Díaz-de Thelin **[DT94]** where the weak solutions are assumed such that $b(u)_t \in L^1(Q)$.

Based in the works **[Vo67]**, **[VoHu69]** and **[J92]**, we shall prove in this paper a comparison result in a class of weak solutions such that $b(u)_t$ is a bounded Radon measure on $Q$ (i.e. $b(u)_t \in \mathcal{M}_b(Q)$). A preliminar version of our comparison result was shortly presented in **[DPa93]**. The present version also contains an enlarged presentation of Chapter 2 of the Ph. D. of the second author (**[Pa95]**). In **[GM92a]**, **[GM92b]**, and **[BeGa95]** the authors prove some comparison results in the class of weak solutions such that $b(u) \in BV(0,T; L^1(\Omega))$ for some related nonlinear parabolic problems but always assuming $p = 2$. Recently, using some techniques raised by S. N. Kruzhkov for hyperbolic equations and inspired in Carrillo **[C86]**, Gagneux and Madaune-Tort proved in **[GM94]** and **[GM95]** a uniqueness result for case $p = 2$. Some more general results for the case $p = 2$ avoiding the assumption $b(u)_t \in L^1(Q)$ can be found in **[P95]**, **[PG96]** and **[U96]** where the authors use that any weak solution satisfies the equation in a "renormalized way".

In Section 2, we introduce the assumptions on the data and the notion of bounded BV solution. Section 3 is devoted to recall several properties on bounded variation functions which will be important for the study of the uniqueness of BV solutions presented later in Section 4. Our main result, a comparison criterium depending on the initial data and the forcing terms, assume a condition on the Hausdorff measure of the set of points where the solutions are not approximately continuous. Finally, the existence of a BV solution is established in Section 5 under some extra information on $u_0$ and $f$.

## 2. Definition of BV solutions

Given a general Banach space $B$, its dual topological space will be denoted by $B'$. By $\langle\cdot,\cdot\rangle_{B,B'}$ we denote the duality between $B'$ and $B$. We shall use the Sobolev space $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$ where $p>1$ and $\dfrac{1}{p}+\dfrac{1}{p'}=1$. Introducing the space $X=L^{p'}(0,T;W^{-1,p'}(\Omega))+\mathrm{L}^1(Q)$ then $X'=L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ and the duality $\langle\cdot,\cdot\rangle_{X,X'}$ is given by

$$\langle h,v\rangle_{X,X'}=\int_0^T\left\{\langle h_{-1},v\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}+\int_\Omega h_1 v\,dx\right\}dt$$

for all $h\in X$, $v\in X'$, where $h=h_{-1}+h_1$ with $h_1\in\mathrm{L}^1(Q)$ and $h_{-1}\in L^{p'}(0,T;W^{-1,p'}(\Omega))$.

We define the space of bounded variation (with respect to variable $t$) functions by

$$BV_t(Q)=\left\{u\in L^1(Q):\frac{\partial u}{\partial t}\in\mathcal{M}_b(Q)\right\},$$

where $\mathcal{M}_b(Q)$ denote the space of bounded Radon measures over $Q$.

In what follows we shall assume a series of conditions on the data:

$$\text{(2.1)}\quad\begin{cases} k:\mathbb{R}\to\mathbb{R},\ \text{is continuous, } k(b(\cdot)) \text{ is Hölder continuous}\\ \quad\text{with exponent } \gamma,\ |k(b(s_1))-k(b(s_2))|\le\hat{C}|s_1-s_2|^\gamma\\ \quad\forall s_1,s_2\in\mathbb{R},\ \text{and } \gamma\ge\frac{1}{p} \text{ if } 1<p\le 2,\ \gamma\ge\frac{1}{2} \text{ if } p>2;\end{cases}$$

$$\text{(2.2)}\quad\begin{cases} g:\Omega\times\mathbb{R}\to\mathbb{R} \text{ is a Caratheodory function such that}\\ \quad|g(x,s)|\le\omega(|s|)(1+d(x)) \text{ for some } d\in L^1(\Omega)\\ \quad\text{and some increasing continuous function } \omega;\end{cases}$$

$$\text{(2.3)}\quad\begin{cases} g(x,s_1)-g(x,s_2)\ge -C^*(b(s_1)-b(s_2))\quad\forall s_1,s_2\in\mathbb{R},\\ s_1>s_2 \text{ and for some } C^*>0;\end{cases}$$

(2.4) $f\in L^1(Q)$

and finally

$$u_0\in L^\infty(\Omega). \tag{2.5}$$

We start by introducing the notion of weak solution of problem (1.1), (1.2) and (1.3) inspired in [**AL83**] and [**DT94**]:

**Definition 1.** We shall say that a function $u$ defined on $Q$ is a *weak solution* of problem (1.1), (1.2) and (1.3), if

$$u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q) \tag{2.6}$$

$$b(u)_t \in L^{p'}(0,T; W^{-1,p'}(\Omega)) + L^1(Q) \tag{2.7}$$

and moreover $u$ satisfies the equality

$$\langle b(u)_t, v\rangle_{X,X'} + \int_0^T \int_\Omega (b(u) - b(u_0))v_t = 0 \tag{2.8}$$

for all $v \in L^p(0,T; W^{1,p}(\Omega)) \cap L^\infty(Q) \cap W^{1,1}(0,T; W^{1,p}(\Omega))$, with $v(T,\cdot) = 0$ and

$$\begin{aligned} \langle b(u)_t, v\rangle_{X,X'} &+ \int_0^T \int_\Omega \phi(\nabla u - k(b(u))\boldsymbol{e}) \cdot \nabla v \\ &+ \int_0^T \int_\Omega g(\cdot, u)v = \int_0^T \int_\Omega fv \end{aligned} \tag{2.9}$$

for all $v \in L^p(0,T; W^{1,p}(\Omega)) \cap L^\infty(Q)$.

By assuming more regularity on $b(u)_t$, we arrive to the following notion:

**Definition 2.** Let $u$ be a weak solution of problem (1.1), (1.2) and (1.3). We shall say that $u$ is a *BV solution* of (1.1), (1.2) and (1.3) if in addition

$$b(u) \in BV_t(Q). \tag{2.10}$$

Notice that in that case $b(u)_t$ is a bounded Radon measure on $Q$ and so, the duality between the spaces $X$ and $X'$ can be also represented by the correspondent integral with respect to the measure $b(u)_t$ for all measurable Borel function $v \in X'$, i.e.

$$\langle b(u)_t, v\rangle_{X,X'} \equiv \int_Q v b(u)_t. \tag{2.11}$$

In what follows, we shall adopt the integral representation of this duality if the test function is a measurable Borel function.

## 3. Treatment of discontinuous functions. Some technical lemmas.

In the development of next sections we shall need some properties of functions whose first generalized derivatives are bounded regular (signed) measures. The following notions and properties can be found in [**Vo67**], [**VoHu85**], [**F69**] and [**EvG92**]. Here, and in what follows, $\mathcal{L}^d(E)$ will denote the $d$-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^d$ and $\mathcal{H}^d(E)$ its $d$-dimensional Hausdorff measure ($d \geq 1$). Let $E$, $F$ be two Lebesgue measurable subsets of $\mathbb{R}^d$. A point $x_0 \in \mathbb{R}^d$ is a *point of $F$-density of the set $E$* if

$$\lim_{r \to 0} \frac{\mathcal{L}^d(B_r(x_0) \cap E \cap F)}{\mathcal{L}^d(B_r(x_0) \cap E)} = 1$$

where $B_r(x_0) = \{x \in R^d : |x - x_0| < r\}$. If the above limit it is equal to 0, the point $x_0$ is a *point of $F$-rarefaction of the set $E$*. Taking $F$ as $\mathbb{R}^d$, we denote by $E_*$ the set of points of density of $E$ and $E^*$ the set of points of rarefaction of $E$. Finally, the set $\partial E = E^* \setminus E_*$ is called the *essential boundary of the set $E$* (in many cases, the essential boundary of a set $E$ coincides with the boundary of $E$, however the boundary and the essential boundary of a set do not always coincide —for example the boundary and essential boundary of a disk minus a radius are not the same [**VoHu85**]). Let now $f : \mathbb{R}^d \to \mathbb{R}$ be a Lebesgue measurable function. The real number $\ell$ is called an *approximate limit* with respect to the set $E \subset \mathbb{R}^d$ of the function $f$ as $x \to x_0$ if for all $\varepsilon > 0$ the point $x_0$ is a point of $E$-density of the set $\{x \in \mathbb{R}^d : |f(x) - \ell| < \varepsilon\}$. We use the notation $\lim_{x \to x_0, x \in E} f(x) = \ell$. Function $f$ is *approximately continuous* at $x_0$ if $\lim_{x \to x_0, x \in \mathbb{R}^d} f(x) = f(x_0)$. A point $x_0$ is called a *regular point* of the function $f$ if there exists an unit vector $\boldsymbol{v}$ such that the approximate limits $f_{\boldsymbol{v}}(x_0)$ and $f_{-\boldsymbol{v}}(x_0)$ exist, where we have denoted $f_{\boldsymbol{v}}(x_0) := \lim_{x \to x_0, \Pi\ (x_0)} f(x)$ with $\Pi_{\boldsymbol{v}}(x_0) = \{x \in R^d : \langle x - x_0, \boldsymbol{v}\rangle > 0\}$. We choose $\boldsymbol{v}$ such that $f_{\boldsymbol{v}}(x_0) \geq f_{-\boldsymbol{v}}(x_0)$. Such a vector $\boldsymbol{v}$ is called a *defining vector*. Vol'pert proved in [**Vo67**] that if $x_0$ is a regular point for $f(x)$ and if $\boldsymbol{v}$ is the defining vector for which $f_{\boldsymbol{v}}(x_0) = f_{-\boldsymbol{v}}(x_0)$, then the associate approximate limit of $f$ in $x_0$ exists and for any $\omega \in \mathbb{R}^d$ $f_\omega(x_0)$ also exists and it is equal to $f(x_0)$. A point verifying this, is called a *point of approximate continuity*. When $f_{\boldsymbol{v}}(x_0) \neq f_{-\boldsymbol{v}}(x_0)$ the vector $\boldsymbol{v}$ is uniquely determined (except for the sign of $f_{\boldsymbol{v}}(x_0)$). A point $x_0$ verifying this inequality is called a *jump point of $f$ in the direction $\boldsymbol{v}$*. The set of jump points of a function $f$ is denoted by $\Gamma_f$. From Theorem 9.2 of [**Vo67**] follows that if $f \in BV(G)$, $G \subset \mathbb{R}^d$, then any point of the $G$ is either a point of approximate continuity or a jump point of $f$ with exception of a set $\mathcal{H}^{d-1}$-dimensional measure zero. For this class of functions, the inward and outward traces of the function $f$ exist on $\Gamma_f$ $\mathcal{H}^{d-1}$-almost everywhere (see e.g. [**VoHu85**]). Moreover these traces coincide with the approximate limits $f_{\boldsymbol{v}}$ and $f_{-\boldsymbol{v}}$ respectively and the defining vector $\boldsymbol{v}$ is the outward normal at the point $x_0$ of $\Gamma_f$.

To extend the differentiation formulas and Green's formula to the class of BV functions it is necessary to define a certain borelian function $\bar{f}$ $\mathcal{H}^{d-1}$-almost everywhere equal to a given function $f$. This borelian representant is the so called *symmetric mean value of $f$*. Let us indicate its connection with the inward and outward traces of $f$, and consequently with the approximate limits $f_{\boldsymbol{v}}$ and $f_{-\boldsymbol{v}}$. We define $\bar{f}$ by the limit (when it exists)

$$\bar{f}(x_0) = \lim_{\varepsilon \to 0} \eta_\varepsilon * f(x_0) \quad (x_0 \in G)$$

where the sequence $\{\eta_\varepsilon\}_\varepsilon$ corresponds to an averaging kernel (see [**VoHu85**, Ch. 4, Section 5, Section 6, p. 181]). It can be shown that if $x_0$ is a regular point of this function, the above limit exists and does not depends on the averaging kernel. Besides at this point the equality

$$\bar{f}(x_0) = \frac{1}{2}[f_{\boldsymbol{v}}(x_0) + f_{-\boldsymbol{v}}(x_0)]$$

holds, where $\boldsymbol{v}$ is a defining vector. In particular, if $\alpha$ is a real continuous function, we can define the *functional superposition* by means of

$$\bar{\alpha}(f(x)) = \int_0^1 \alpha(f_{\boldsymbol{v}}(x))s + \alpha(f_{-\boldsymbol{v}}(x))(1-s)\, ds.$$

**Remark 1.** An important property is that $\bar{\alpha}(f(x_0)) = \alpha(f(x_0))$ for any $x_0$ point of approximate continuity of $f$. Since any summable function $f$ is approximately continuous $\mathcal{L}^d$-almost everywhere, then the above equality holds $\mathcal{L}^d$-almost everywhere. So, if $f$ is $\mathcal{H}^{d-1}$-approximately continuous function, then $\bar{\alpha}(f) = \alpha(f)$ $\mathcal{H}^{d-1}$-almost every where in $G$.

However, $d$-dimensional measure is too large when we try to apply differentiation formulas to functions with measures as generalized derivatives. The generalizes of the classical formulas of differentiation by using the symmetric mean value to functions with measures as generalized derivatives are shown in [**VoHu85**, Chapter 5, Section 1].

By applying these notions to the case of $G = Q$ and $d = N+1$, we can obtain the following lemma which gives an important property of the functions whose generalized derivatives are summable functions.

**Lemma 1.** *Let $u \in W^{1,1}(Q)$. Then, $u$ is $\mathcal{H}^N$-almost everywhere approximately continuous on $Q$, i.e. $\mathcal{H}^N(\Gamma_u) = 0$.*

*Proof:* If $u \in W^{1,1}(Q)$ it is clear that $u \in BV(Q)$. Then there exists $\Lambda \subset Q$, with $\mathcal{H}^N(\Lambda) = 0$ and such that $Q - \Lambda = \{(t,x) \in Q$: regular points$\}$. So, for any $(t,x)$ in $\Gamma_u - \Lambda$, there exists an unique vector $\boldsymbol{v} = (\boldsymbol{v}_t, \boldsymbol{v}_x)$ (depending of the point $(t,x)$ and where $\boldsymbol{v}_t \in \mathbb{R}$ and $\boldsymbol{v}_x \in \mathbb{R}^N$) which is the inward normal and there exist the approximated limits $u_{\boldsymbol{v}}(t,x)$ and $u_{-\boldsymbol{v}}(t,x)$ (see [**VoHu85**]). Let $S$ be a Borel subset of $\Gamma_u - \Lambda$. Since $u \in W^{1,1}(Q)$, one has that $\mathcal{L}^{N+1}(S) = 0$ (see [**Vo67**], [**VoHu69**]). From that, and as $u \in W^{1,1}(Q)$, it follows that

$$\int_S \frac{\partial u}{\partial x_i} = \int_Q \chi_S \frac{\partial u}{\partial x_i}\, dx\, dt = 0, \quad (i = 1, 2, 3, \dots, N)$$
$$\int_S \frac{\partial u}{\partial t} = \int_Q \chi_S \frac{\partial u}{\partial t}\, dx\, dt = 0.$$

($\chi_S$ is the characteristic function of the subset $S$). Applying now Theorem 2, p. 203 of [**VoHu85**] we get

$$0 = \int_S (u_{\boldsymbol{v}} - u_{-\boldsymbol{v}}) \boldsymbol{v}_{x_i} d\mathcal{H}^N,$$
$$0 = \int_S (u_{\boldsymbol{v}} - u_{-\boldsymbol{v}}) \boldsymbol{v}_t d\mathcal{H}^N \quad (i = 1, 2, 3, \dots, N).$$

The above equality implies that $\mathcal{H}^N(S) = 0$. Finally, as $S$ is arbitrary, we conclude that $\mathcal{H}^N(\Gamma_u) = 0$. ■

The main result of the general theory of BV functions that we shall use later in order to prove our uniqueness theorem is given in the following lemma:

**Lemma 2.** *Let $u \in BV_t(Q)$. Then, the measure $u_t$ is $\mathcal{H}^N$-absolutely continuous.*

**Remark 2.** If we assume that $u \in BV_t(Q)$ and $v \in L^\infty(Q)$ $\mathcal{H}^N$-approximately continuous function on $Q$, then by the above lemma and Remark 1 we have that $\bar{v} = v$ $\dfrac{\partial u}{\partial t}$-almost every where on $Q$.

*Proof of Lemma 2:* Let $A$ be a borelian subset of $Q$. Let $A_\Omega$ be the projection of $A$ over the hyperplane $\{t = 0\}$. If $\mathcal{H}^N(A) = 0$ then $\mathcal{L}^N(A_\Omega) = 0$ (see Vol'pert [**Vo67**]). For a fixed $x \in \Omega$, we denote by $\operatorname{ess} V_0^T(u^{(x)})$ the essential variation of $u^{(x)}(t) := u(t,x)$ as the function of

$t \in (0,T)$ given by $\equiv \sup\{\sum |u(t_i,x) - u(t_{i-1},x)|\}$. Since $u \in BV_t(Q)$, $\operatorname{ess} V_0^T(u^{(x)})$ is defined almost everywhere $x \in \Omega$ with respect to the Lebesgue measure and $\mathcal{L}^N$ summable in $A_\Omega$ (see [**Vo67**] and [**EvG92**]). Moreover

$$\left|\frac{\partial u}{\partial t}\right|(]0,T[\times A_\Omega) = \int_{\Omega_A} \operatorname{ess} V_0^T(u^{(x)})\,dx = 0$$

since $\mathcal{L}^N(A_\Omega) = 0$ and so the statement of the lemma holds. ■

**Lemma 3.** *Assume $u \in L^\infty(Q)$ and $b$ as in (1.5). If in addition we assume that*

$$b(u) \in BV_t(Q), \tag{3.1}$$

*and*

$$b^{-1} \textit{ locally Lipschitz on } [-\|u\|_{L^\infty(Q)}, \|u\|_{L^\infty(Q)}] \tag{3.2}$$

*then*

$$u \in BV_t(Q).$$

*Proof:* To prove this property, we show that $u_t$ is a bounded Radon measure on $Q$. Following Vol'pert and Hudajaev [**VoHu85**, Chapter 4, Section 2] it is enough to prove that there exists a positive constant $K$ such that $|\langle u_t, \varphi\rangle| \le K\|\varphi\|_{L^\infty(Q)}$ for all $\varphi \in C_c^1(Q)$. In order to do that, we use the fact that $u_t = \lim_{h\to 0} \dfrac{u(t+h,x) - u(t,x)}{h}$ in sense of distributions. From the assumptions (3.1) and (3.2) we obtain the result. ■

**Remark 3.** Condition (3.2) sometimes is verified in an implicit way. For instance, if

$$\begin{cases} b = \lambda_1 b_1 + \lambda_2 b_2, \text{ with } b_1, b_2 \text{ continuous functions} \\ \quad \lambda_1, \lambda_2 \ge 0, \text{ and } b_1^{-1}, b_2 \text{ locally Lipschitz;} \end{cases} \tag{3.3}$$

and

$$\lambda_1 - L_1 L_2 \lambda_2 > 0 \tag{3.4}$$

where $L_1$ and $L_2$ are the Lipschitz constant of $b_1^{-1}$ and $b_2$ respectively on the interval $[-\|u\|_{L^\infty(Q)}, \|u\|_{L^\infty(Q)}]$, then necessarily $b^{-1}$ is Lipschitz on the interval $[-\|u\|_{L^\infty(Q)}, \|u\|_{L^\infty(Q)}]$.

**Lemma 4.** *Let be $u \in BV_t(Q) \cap L^\infty(Q)$ and let be $\eta$ a locally bounded borelian function in $\mathbb{R}$. Define the function $H$ given by*

$$H(r) = \int_0^r \eta(s)\, ds,$$

*for all $r \in \mathbb{R}$. Then:*

a) *$H(u)$ belongs to $BV_t(Q) \cap L^\infty(Q)$;*

b) *the following relation*

$$\int_{Q_t} \phi \frac{\partial H(u)}{\partial s} = \int_{Q_t} \phi \bar{\eta}(u) \frac{\partial u}{\partial s}$$

*holds for all $\phi$ bounded borelian function on $Q_t = ]0, t[\times\Omega$. In particular, we have the following "chain rule formula"*

$$\frac{\partial H(u)}{\partial t} = \bar{\eta}(u) \frac{\partial u}{\partial t}$$

*in measure sense;*

c) *for any $v \in BV_t(Q) \cap L^\infty(Q)$, we have*

$$\int_{Q_t} \bar{H}(u) \frac{\partial v}{\partial s} = \int_\Omega \bar{H}(u)(t,x)\bar{v}(t,x)\, dx - \int_\Omega \bar{H}(u)(0,x)\bar{v}(0,x)\, dx - \int_{Q_t} \bar{v}\bar{\eta}(u)\frac{\partial u}{\partial s};$$

d) *if in addition, $\eta$ is a real continuous function, $u$ and $v$ are $\mathcal{H}^N$-approximately continuous functions (i.e., $\mathcal{H}^N(\Gamma_u) = 0 = \mathcal{H}^N(\Gamma_v)$) then the relations given in b) and c) are also true replacing $\bar{\eta}(u)$, $\bar{H}(u)$ and $\bar{v}$ by $\eta(u)$, $H(u)$ and $v$ respectively.*

*Proof:* a) since $H$ is a locally Lipschitz continuous function the conclusion comes from Lemma 3. b) is consequence of a), the rule chain for the one-dimensional case (Theorem 13.2 of [**Vo67**]) and Theorem 4.5.9 of [**F69**]. c) is proved using the integration by parts formula for BV function [**VoHu85**] and the above mentioned theorem of [**F69**]. Finally, we obtain d). From the fact that $u$ and $v$ are $\mathcal{H}^N$-absolutely continuous functions and the properties of functional superposition we obtain that the borelian representatives $\bar{\eta}(u)$, $\bar{H}(u)$ and $\bar{v}$ are equal to the functions $\eta(u)$, $H(u)$ and $v$ $\mathcal{H}^N$-almost everywhere where in $Q_t$ respectively (see Remark 1). And finally, applying Lemma 2 (see Remark 2) we conclude the proof. ∎

**Remark 4.** Notice that if $u \in BV_t(Q) \cap L^\infty(Q)$ and $H \in C^1(\mathbb{R})$, then, Lemma 4 implies $H(u) \in BV_t(Q) \cap L^\infty(Q)$. If in addition $u$ is $\mathcal{H}^N$-approximately continuous function on $Q$ then $H(u)_t = H'(u)u_t$.

## 4. Comparison and continuous dependence results

In this section, we give several results on the comparison and continuous dependence of BV solutions of the problem (1.1), (1.2) and (1.3) under the main condition (4.3). We shall use later the inequality

$$(4.1)\quad |\phi(\eta)-\phi(\hat{\eta})|^{p'} \leq C\left\{[\phi(\eta)-\phi(\hat{\eta})]\cdot[\eta-\hat{\eta}]\right\}^{\beta/2}\left\{|\eta|^p+|\hat{\eta}|^p\right\}^{1-\frac{\beta}{2}}$$

where $\beta=2$ if $1<p\leq 2$ and $\beta=p'$ if $p\geq 2$ which holds for any $\eta$ and $\hat{\eta}$ in $\mathbb{R}^N$ from Tartar's inequality (see e.g. Díaz-de Thelin [**DT94**]).

**Theorem 1.** *Assume that $b$, $k$ and $g$ verify (1.5), (2.1), (2.2) and (2.3). Let $(f_1, u_{0_1})$ and $(f_2, u_{0_2})$ be a pair of data satisfying (2.4) and (2.5). Let $u_1$ and $u_2$ be two BV solutions of the problem (1.1), (1.2) and (1.3) associated to $(f_1, u_{0_1})$ and $(f_2, u_{0_2})$ respectively. We also suppose that*

$$(4.2)\qquad u_1 \text{ and } u_2 \in BV_t(Q)$$

*and that*

$$(4.3)\qquad \mathcal{H}^N(\Gamma_{u_1})=\mathcal{H}^N(\Gamma_{u_2})=0.$$

*Then, for any $t\in[0,T]$ we have*

$$\int_\Omega [b(u_1(t,x))-b(u_2(t,x))]_+\,dx \leq e^{C^*t}\left\{\int_\Omega [b(u_{0_1}(x))-b(u_{0_2}(x))]_+\,dx + \int_0^t\int_\Omega e^{-C^*s}[f_1(s,x)-f_2(s,x)]_+\,dx\,ds\right\}.$$

**Remark 5.** i) The regularity (4.2) on the functions $u_i$ can be obtained by assuming some regularity properties on function $b$. In particular we note that if $b^{-1}$ is a locally Lipschitz continuous function, condition $b(u_i)\in BV_t(Q)$ implies (4.2) (see Lemma 3). ii) Also, we can assume $b$ as in (3.3) and if $M$ is a positive constant such that $\|u_i\|_{L^\infty(Q)}\leq M$, for $i=1,2$ and we suppose that $b_1^{-1}$ and $b_2$ have Lipschitz constants $L_1$ and $L_2$ respectively on the interval $[-M,M]$ with

$$(4.4)\qquad \lambda_1 - L_1L_2\lambda_2 > 0,$$

then (4.2) holds (see the Remark 3 and Lemma 3).

**Remark 6.** The case $b$ locally Lipschitz continuous function was previously considered in [**DT94**].

**Remark 7.** If $u_t \in L^1(Q)$ then $u \in BV_t(Q)$ and assumption (4.4) is not needed. Notice that in that case assumption (4.3) always holds due to Lemma 1.

Some consequences of the above theorem are the following results:

**Corollary 1.** *Let $u_1$ and $u_2$ be two BV solutions as in Theorem 1 associated to the data $(f_1, u_{0_1})$ and $(f_2, u_{0_2})$. Assume that $f_1 \leq f_2$ and $u_{0_1} \leq u_{0_2}$. Then $u_1 \leq u_2$ in $Q$.*

*Proof:* Since $f_1 \leq f_2$ and $u_{0_1} \leq u_{0_2}$, then $[f_1 - f_2]_+ = 0$ and $[u_{0_1} - u_{0_2}]_+ = 0$ respectively. Applying the above theorem, we obtain

$$\int_\Omega [b(u_1(t,x)) - b(u_2(t,x))]_+ \, dx \leq 0$$

and so $u_1 \leq u_2$ thanks to the monotonicity of function $b$. ■

**Corollary 2.** *If $u_1$ and $u_2$ are two BV solutions like in Theorem 1, for any $t \in [0,T]$ we have*

$$\|b(u_1(t,\cdot)) - b(u_2(t,\cdot))\|_{L^1(\Omega)} \leq e^{C^* t} \left\{ \|b(u_{0_1}) - b(u_{0_2})\|_{L^1(\Omega)} + \int_0^t e^{-C^* s} \|f_1 - f_2\|_{L^1(\Omega)} \, ds \right\}. \ ■$$

*Proof:* It suffices to recall that $\|\cdot\|_{L^1(\Omega)} = |(\cdot)_+|_{L^1(\Omega)} + |(\cdot)_-|_{L^1(\Omega)}$ and to apply Theorem 1 (notice that given $s \in \mathbb{R}$, we call $s_- = \max\{0, -s\} = (-s)_+$ and then $|s| = s_+ + s_- = s_+ + (-s)_+$). ■

Finally, we obtain the uniqueness of BV solutions in the class of functions given in Theorem 1:

**Corollary 3.** *At most there exits one BV solution $u$ of (1.1), (1.2) and (1.3) under the assumptions (1.5), (2.1), (2.2), (2.3), (2.4) and (2.5), in the class of solutions verifying (4.2) and (4.3).*

*Proof:* Take $f_1 = f_2$ and $u_{0_1} = u_{0_2}$ in Corollary 2. ■

Notice that the above results are also true under the conditions of Remark 5 given in the case i) and in the case ii).

**Corollary 4.** *Assume that $b$, $k$ and $g$ verify (1.5), (3.3), (2.1), (2.2) and (2.3). Let $(f_1, u_{0_1})$ and $(f_2, u_{0_2})$ be a pair of data satisfying (2.4) and (2.5). Let $u_1$ and $u_2$ be two BV solutions of the problem (1.1), (1.2) and (1.3) associated to $(f_1, u_{0_1})$ and $(f_2, u_{0_2})$ respectively. Also suppose that $\|u_i\|_{L^\infty(Q)} \leq M$ with $M > 0$ for $i = 1, 2$, and suppose that $b_1^{-1}$ and $b_2$ have Lipschitz constants $L_1$ and $L_2$ respectively in the interval $[-M, M]$ satisfying (4.4). Finally, we also assume (4.3). Then, for any $t \in ]0, T[$ we have that*

$$\int_\Omega [b(u_1(t,x)) - b(u_2(t,x))]_+ \, dx \leq e^{C^* t} \left\{ \int_\Omega [b(u_{0_1}(x)) - b(u_{0_2}(x))]_+ \, dx + \int_0^t \int_\Omega e^{-C^* s} [f_1(s,x) - f_2(s,x)]_+ \, dx \, ds \right\}.$$

*Proof:* Under assumption (4.4), we obtain that $u_1$ and $u_2 \in BV_t(Q)$ from Lemma 3. ■

Arguing as before, we can obtain analogous results to Corollaries 1-3 for BV solutions which lie in $[-M, M]$. On the other hand, we can make explicit $M$ for bounded data

**Lemma 5.** *Let $u$ be a weak solution of (1.1), (1.2) and (1.3). Assume (1.5), (2.1), (2.2), (2.3), and for the data, we assume $u_0 \in L^\infty(\Omega)$ and $f \in L^1(0, T; L^\infty(\Omega))$. Then*

$$\|b(u)\|_{L^\infty(Q)} \leq e^{C^* T} \left\{ \|b(u_0)\|_{L^\infty(\Omega} + \int_0^T e^{-C^* s} \|f(s, \cdot)\|_{L^\infty(\Omega)} \, ds \right\}.$$

*Thus, there exists a positive constant $M > 0$ such that*

$$\|u\|_{L^\infty(Q)} \leq M.$$

*Proof:* See e.g. Benilan [**Be81**]. ■

Thanks to Lemma 5 and Corollary 4, we have

**Corollary 5.** *Assume $u_0 \in L^\infty(\Omega)$ and $f \in L^1(0, T; L^\infty(\Omega))$. Let $M > 0$ given by Lemma 5. Assume also (1.5), (3.3), (2.1), (2.2), (2.3) and (4.3). Then, there exists at most one BV solution of (1.1), (1.2) and (1.3).*

*Proof of Theorem 1:* For any $n \in \mathbb{N}$, we define $T_n$, approximation of the $\text{sign}_+^0$ function ($\text{sign}_+^0(s) := -1$ if $s < 0$, $0$ if $s = 0$, $1$ if $s > 0$), by

$$T_n(s) = \begin{cases} 0 & s \leq 0, \\ \dfrac{n^2 s^2}{2} & 0 < s \leq \dfrac{1}{n}, \\ 2ns - \dfrac{n^2 s^2}{2} - 1 & \dfrac{1}{n} < s \leq \dfrac{2}{n}, \\ 1 & s > \dfrac{2}{n}. \end{cases}$$

It is easy to see that

$$\text{(4.5)} \qquad \begin{cases} 0 \leq T_n'(s) \leq n, \ \lim_{n\to\infty} sT_n'(s) = 0, \\ |T_n(s)| \leq 1, \ \lim_{n\to\infty} T_n(s) = \text{sign}_+(s) \quad \text{and} \\ \lim_{n\to\infty} sT_n(s) = s_+ = \begin{cases} 0 & s \leq 0 \\ s & s > 0. \end{cases} \end{cases}$$

To simplify the notation, we set $z = b(u_1) - b(u_2)$ and $\xi_1 = \nabla u_1 - k(b(u_1))e$, $\xi_2 = \nabla u_2 - k(b(u_2))e$. We have that $T_n(u_1 - u_2) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap \mathrm{L}^\infty(Q)$ is an admissible test functions since $u_1$ and $u_2$ are BV solutions and $T_n$ is a regular function. As moreover, we are assuming (4.3), then $\bar{T}_n(u_1 - u_2) = T_n(u_1 - u_2)$ $\mathcal{H}^N$-a.e. in $Q$. Thus, and thanks to Lemma 2, we can adopt the notation (2.11); that is

$$\begin{aligned} \langle b(u_i)_t, T_n(u_1 - u_2)\rangle_{X,X'} &= \int_{Q_t} \bar{T}_n(u_1 - u_2) b(u_i)_t \\ &= \int_{Q_t} T_n(u_1 - u_2) b(u_i)_t \quad i = 1, 2. \end{aligned}$$

Considering the relations (2.9) verified by $u_1$ and $u_2$ and subtracting, we obtain

$$\begin{aligned} -\int_{Q_t} T_n(u_1 - u_2) z_t = &\int_{Q_t} [\phi(\xi_1) - \phi(\xi_2)] \cdot \nabla T_n(u_1 - u_2)\, dx\, dt \\ &+ \int_{Q_t} (g(x, u_1) - g(x, u_2)) T_n(u_1 - u_2)\, dx\, dt \\ &- \int_{Q_t} (f_1(t,x) - f_2(t,x)) T_n(u_1 - u_2)\, dx\, dt \end{aligned}$$

where $Q_t = ]0, t[\times\Omega$, $(0 < t < T)$.

In order to pass to the limit we need some technical results

**Lemma 6.** *Under the assumptions of Theorem 1, we have*

$$\lim_{n\to\infty}\int_0^t\int_\Omega T_n(u_1-u_2)\frac{\partial(b(u_1)-b(u_2))}{\partial s} = \int_\Omega [b(u_1)-b(u_2)]_+(t)\,dx - \int_\Omega [b(u_{0_1})-b(u_{0_2})]_+\,dx$$

*for any* $t\in[0,T]$.

*Proof of Lemma 6:* Since $u_1$ and $u_2$ are BV solutions, we have that $b(u_1)$ and $b(u_2)$ belong to $BV_t(Q)\cap L^\infty(Q)$. On the other hand, by Lemma 4 and Remark 4, $T_n(u_1-u_2)\in BV_t(Q)\cap L^\infty(Q)$. Moreover by assumption (4.3), for all $n\in N$, $T_n(u_1-u_2)$ is also an $\mathcal{H}^N$-approximately continuous function (see [**VoHu85**, Theorem 2, p. 164]). Thus, using that $b$ is strictly increasing

$$T_n(u_1-u_2)\xrightarrow[n\to\infty]{}\mathrm{sign}_+^0(u_1-u_2) = \mathrm{sign}_+^0(b(u_1)-b(u_2))\,\mathcal{H}^N\text{—a.e. in } Q. \tag{4.6}$$

Applying the Lebesgue's theorem with measure $\frac{\partial z}{\partial t}$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_{Q_t} T_n(u_1-u_2)\frac{\partial z}{\partial s} &= \int_{Q_t}\mathrm{sign}_+(u_1-u_2)\frac{\partial z}{\partial s}\\ &= \int_{Q_t}\mathrm{sign}_+(z)\frac{\partial z}{\partial s}\\ &= \lim_{n\to\infty}\int_{Q_t} T_n(z)\frac{\partial z}{\partial s}\end{aligned} \tag{4.7}$$

since, by Lemma 2, $\frac{\partial z}{\partial t}$ is $\mathcal{H}^N$-absolutely continuous. By part c) and d) of Lemma 4 we have

$$\int_{Q_t} T_n(z)\frac{\partial z}{\partial s} = \int_\Omega T_n(z)(t)z(t)\,dx - \int_\Omega T_n(z)(0)z(0)\,dx - \int_{Q_t} zT_n'(z)\frac{\partial z}{\partial s}$$

and passing to the limit we get

$$\lim_{n\to\infty}\int_{Q_t} T_n(z)\frac{\partial z}{\partial s} = \int_\Omega \mathrm{sign}_+(z(t))z(t)\,dx - \int_\Omega \mathrm{sign}_+(z(0))z(0)\,dx \tag{4.8}$$

from Lebesgue's theorem and the conclusion holds. ■

**Lemma 7.** *Under the assumption of Theorem 1 we have*

$$\lim_{n\to\infty}\int_\Omega T_n'(u_1(t)-u_2(t))[\phi(\xi_1(t))-\phi(\xi_2(t))] \cdot [\nabla u_1(t)-\nabla u_2(t)]\,dx \geq 0 \tag{4.9}$$

*for a.e.* $t\in]0,T[$, *i.e. the diffusion operator is* $T$*-acretive in* $L^1(\Omega)$.

*Proof of Lemma 7:* It is a slight improvement of Díaz-de Thelin [**DT94**]. For the sake of completeness we give the detailed proof. For any $n\in\mathbb{N}$, the integral term in (4.9) it can be written as the addition of the integrals

$$I_1(n)=\int_\Omega T_n'(u_1-u_2)[\phi(\xi_1)-\phi(\xi_2)]\cdot[\xi_1-\xi_2]\,dx$$

and

$$I_2(n)=\int_\Omega T_n'[\phi(\xi_1)-\phi(\xi_2)]\cdot e(k(b(u_1))-k(b(u_2)))\,dx.$$

Here we drop writing the $t$-dependence. We shall find an estimate on $|I_2(n)|$ in terms of $I_1(n)$. Due to the assumption (2.1) on $k$, we need to distinguish the cases $1<p\leq 2$ and $p>2$.

*Case* $1<p\leq 2$*:* Applying Young's inequality we get

$$|I_2(n)|\leq\frac{\varepsilon}{p'}\int_\Omega T_n'|\phi(\xi_1)-\phi(\xi_2)|^{p'} + \frac{1}{p\varepsilon}\int_\Omega T_n'|k(b(u_1))-k(b(u_2))|^p \tag{4.10}$$

for any $\varepsilon>0$. Using the inequality (4.1) to the first term of the right hand side of (4.10), by assumption (2.1) on $k$ and the properties of $T_n'$ we obtain that

$$|I_2(n)|\leq\frac{\varepsilon^{p'}C}{p'}I_1(n)+\frac{2\hat{C}}{\varepsilon^p p}\left(\frac{2}{n}\right)^{\gamma p-1}\mathcal{L}^N\left(\left\{x:0<u_1-u_2<\frac{2}{n}\right\}\right).$$

Taking $\varepsilon^{p'}=p'/C$, we get

$$-\frac{2\hat{C}C^{p-1}}{pp'}\left(\frac{2}{n}\right)^{\gamma p-1}\mathcal{L}^N\left(\left\{x:0<u_1-u_2<\frac{2}{n}\right\}\right)\leq I_1(n)+I_2(n).$$

Now, since $\gamma p - 1 > 0$, we have that

$$\lim_{n\to\infty} I_1(n) + I_2(n) \geq 0$$

and then (4.9) is proved.

*Case $p > 2$:* By Hölder inequality

$$|I_2(n)| \leq \left\{ \int_\Omega T_n' |\phi(\xi_1) - \phi(\xi_2)|^{p'}\, dx \right\}^{\frac{1}{p'}} \cdot \left\{ \int_\Omega T_n' |k(b(u_1)) - k(b(u_2))|^p\, dx \right\}^{\frac{1}{p}}. \tag{4.11}$$

Using inequality (4.1) where we set $\beta = p'$ and $\eta_1 = \xi_1$, $\eta_2 = \xi_2$, the first multiplicative factor in (4.11) is bounded by

$$C^{\frac{1}{p'}} \left\{ \int_\Omega \left\{ T_n' \Big[\phi(\xi_1) - \phi(\xi_2)\Big][\xi_1 - \xi_2] \right\}^{\frac{p'}{2}} \left\{ T_n'(|\xi_1|^p + |\xi_2|^p) \right\}^{1-\frac{p'}{2}} dx \right\}^{\frac{1}{p'}}.$$

Using again the Hölder inequality and the properties of $T_n'$, we obtain the estimate

$$\left\{ \int_\Omega T_n' |\phi(\xi_1) - \phi(\xi_2)|^{p'}\, dx \right\}^{\frac{1}{p'}} \leq \mathcal{A}(n) I_1^{\frac{1}{2}}(n) n^{\frac{2-p'}{2p'}}$$

where

$$\mathcal{A}(n) = C^{\frac{1}{p'}} \left\{ \int_{\Omega \cap \{0 < u_1 - u_2 < \frac{2}{n}\}} |\xi_1|^p + |\xi_2|^p\, dx \right\}^{\frac{2-p'}{2p'}}.$$

For the second multiplicative factor in (4.11), we have

$$\left\{ \int_\Omega T_n' |k(b(u_1)) - k(b(u_2))|^p\, dx \right\}^{1/p} \leq 2^\gamma n^{\frac{1-p\gamma}{p}} \left\{ \mathcal{L}^N \left\{ x : 0 < u_1 - u_2 < \frac{2}{n} \right\} \right\}^{1/p}$$

from assumption (2.1) on $k$ and (4.5). Combining both inequalities, we arrive to

$$|I_2(n)| \leq 2^\gamma \mathcal{A}(n)\, n^{\frac{2-p'}{2p'} + \frac{1-p\gamma}{p}} I_1^{\frac{1}{2}}(n)\, \mathcal{L}^N \left( \left\{ x : 0 < u_1 - u_2 < \frac{2}{n} \right\} \right)^{\frac{1}{p}}.$$

Since $2^\gamma \mathcal{A}(n)\mathcal{L}^N\left(\left\{x : 0 < u_1 - u_2 < \frac{2}{n}\right\}\right)^{1/p}$ is uniformly bounded for any $n \in \mathbb{N}$ and the exponent $\frac{2-p'}{2p'} + \frac{1-p\gamma}{p}$ is positive, due to (2.1), we conclude that

$$\lim_{n\to\infty}(I_1(n) + I_2(n)) \geq 0$$

which ends the proof. ∎

*End of the proof of Theorem 1:* By the previous two lemmas, we obtain the key inequality

$$\begin{aligned}\int_\Omega [b(u_1(t,x)) - b(u_2(t,x))]_+\,dx \leq \int_\Omega [b(u_{0_1}(x)) - b(u_{0_2}(x))]_+\,dx\\ - \int_{Q_t}[g(x,u_1) - g(x,u_2)]\operatorname{sign}_+(u_1 - u_2)\,dx\,ds\\ + \int_{Q_t}[f_1(t,x) - f_2(t,x)]\operatorname{sign}_+(u_1 - u_2)\,dx\,ds.\end{aligned}$$

Using the assumption (2.3) on $g$, the conclusion of the theorem is immediate if $C^*$ is zero. More in general we set $v_j(t,x) = e^{-C^*t}b(u_j(t,x))$ for $j = 1,2$. Then $\operatorname{sign}_+(v_1 - v_2) = \operatorname{sign}_+(u_1 - u_2)$ and $\frac{\partial v_j}{\partial t} = -C^*v_j + e^{-C^*t}\frac{\partial b(u_j)}{\partial t}$ are also bounded regular measures in $Q$. Choosing $T_n(v_1 - v_2)$ as test function and working as before, we obtain

$$\begin{aligned}\int_\Omega [v_1(t,x) - v_2(t,x)]_+\,dx \leq \int_\Omega [v_{0_1}(x) - v_{0_2}(x)]_+\,dx\\ - C^*\int_{Q_t}[v_1(s,x) - v_2(s,x)]_+\,dx\,ds\\ - \int_{Q_t}[g(x,u_1) - g(x,u_2)]e^{-C^*s}\operatorname{sign}_+(v_1(s,x) - v_2(s,x))\,dx\,ds\\ + \int_{Q_t}[f_1(s,x) - f_2(s,x)]e^{-C^*s}\operatorname{sign}_+(v_1(s,x) - v_2(s,x))\,dx\,ds.\end{aligned}$$

By assumption (2.3), one has that

$$\begin{aligned}&-[g(x,u_1) - g(x,u_2)]e^{-C^*s}\operatorname{sign}_+(v_1 - v_2)\\ &\qquad \leq C^*[b(u_1) - b(u_2)]e^{-C^*s}\operatorname{sign}_+(v_1 - v_2) = C^*[v_1 - v_2]_+\end{aligned}$$

and thus, the conclusion holds. ∎

**Remark 8.** The assumption (4.3) on the measure of the jump points set is merely needed in the proof of Lemma 6. This assumption could be replaced by any other condition implying the conclusion of Lemma 6. In particular, we have

**Corollary 6.** *Let $u_1$ and $u_2$ be bounded BV solutions of (1.1), (1.2) and (1.3). Assume the hypotheses of Theorem 1 but replacing (4.3) by*

$$\text{(4.12)} \qquad \begin{cases} \text{there exist two homeomorphisms on } \mathbb{R}, \Psi_1 \text{ and } \Psi_2, \\ \text{such that } \Psi_1(u_1), \Psi_2(u_2) \in W^{1,1}(Q). \end{cases}$$

*Then $u_1$ and $u_2$ verify the comparison criterium given in Theorem 1.*

*Proof:* By Lemma 1 condition (4.3) holds for $u_i$, $i = 1, 2$. ■

## 5. Existence of bounded BV solutions

In order to obtain the existence of bounded BV solutions for problem (1.1), (1.2), (1.3) we shall assume some additional conditions on functions $f$ and $u_0$:

$$\text{(5.1)} \qquad f \in L^\infty(Q) \cap BV_t(Q),$$

and

$$\text{(5.2)} \qquad u_0 \in L^\infty(\Omega) \cap W_0^{1,p} \text{ and } \phi(\nabla u_0 - k(b(u_0))e) \in (BV(\Omega))^N.$$

We state our existence result in the following way:

**Theorem 2.** *Assume (1.5) on $b$ and also that*

$$\text{(5.3)} \qquad b^{-1} \textit{ is a locally Lipschitz continuous function.}$$

*We assume also (2.1), (2.2), (2.3), (5.1) and (5.2). Then there exists a bounded BV solution $u$ of (1.1), (1.2) and (1.3). Moreover $u \in C([0,T]; L^1(\Omega))$.*

*Proof:* We start by considering, a sequence of regular problems having a unique solution by the classical theory of partial differential equations. After that, we shall obtain suitable a priori estimates. Finally passing to the limit we shall find a bounded BV solution. In view of the structural assumptions, we shall distinguish two cases, according $p$ satisfies $1 < p < 2$ or $2 \geq p$.

*Case $1 < p < 2$: Regularization.* We define a sequence of uniformly parabolic problems with coefficients and free term bounded regular functions. Consider the following regularized equation in $Q$

$$\text{(5.4)} \qquad \frac{\partial b_m(u)}{\partial t} - \operatorname{div} \phi_r(\nabla u - k_s(b_m(u))e) + g_n(x,u) = f_l(t,x),$$

where we define the vectorial function $\phi_r$ by

$$\phi_r(\xi) = \begin{cases} \left(|\xi|^2 + \dfrac{1}{r}\right)^{\frac{p-2}{2}} \xi & |\xi| < \dfrac{1}{r}, \\ |\xi|^{p-2}\xi & |\xi| \geq \dfrac{2}{r}, \\ \phi_r(\xi) & \dfrac{1}{r} \leq |\xi| < \dfrac{2}{r}, \end{cases}$$

for any $r \in \mathbb{N}$, and such that $\phi_r \in C^1(\mathbb{R}^N)$ verifies

$$|\phi_r(\xi)| \leq |\xi|^{p-1} \quad \forall \xi \in \mathbb{R}^N \tag{5.5}$$

and

$$\frac{\alpha|\xi|^2}{(1+|\xi|)^{2-p}} \leq \phi_r(\xi) \cdot \xi \leq |\xi|^p \tag{5.6}$$

for any $\xi$ in $\mathbb{R}^N$ $\alpha := 2^{\frac{p-2}{2}}$. For any $m \in \mathbb{N}$, we define

$$b_m(\eta) = \frac{1}{m}\eta + \bar{b}_m(\eta)$$

with $\bar{b}_m$ the Yosida approximation of $b$. We recall that $\bar{b}_m$ converges uniformly on compacts sets to $b$, $\bar{b}_m$ is a Lipschitz nondecreasing function such that $|\bar{b}_m| \leq |b|$ and that $b_m$ and $b_m^{-1}$ are Lipschitz nondecreasing functions; see [**Be81**].

We take a sequence of functions $\{k_s\}_{s=1}^{\infty}$ belonging to $C^\infty(\mathbb{R})$ such that they verify (2.1) and $k_s$ converges to $k$ uniformly on compacts of $\mathbb{R}$.

For any integer $n$, we consider a function $g_n \in C^\infty(\Omega \times \mathbb{R})$ satisfying the assumptions (2.2) and (2.3) uniformly on $n$ and such that $g_n(x,\eta)$ converges to $g_n(x,\eta)$ in $L^1(\Omega)$ for any fixed $\eta$ in $\mathbb{R}$, for a.e. $x \in \Omega$, as $n \to \infty$.

Let $f_l \in C^\infty([0,T] \times \bar{\Omega})$ such that

$$\|f_l\|_{L^\infty(Q)} \leq C\|f\|_{L^\infty(Q)}, \ \|f_l\|_{W^{1,1}(0,T;L^1(\Omega))} \leq C\|f\|_{BV_t} \text{ for any } l \in \mathbb{N}$$

and such that $f_l$ converges to $f$ in $L^1(Q)$ as $l \to \infty$.

Finally we regularize the initial condition. We consider $u_{0,q} \in C_0^\infty(\Omega)$ such that $u_{0,q} \stackrel{*}{\rightharpoonup} u_0$ in $L^\infty(\Omega)$ as $q \to \infty$ and such that $\|\phi_r(\nabla u_{0,q} - k_s(b_m(u_{0,q}))e)\|_{BV(\Omega)^N} \leq \|\phi_r(\nabla u_0 - k(b(u_0))e)\|_{BV(\Omega)^N}$.

The equation (5.4) is uniformly parabolic. So, by well-know result (see e.g. Ladyzenskaja, Solonnikov and Uralceva [**LSU68**, Chap. V]) there exists a unique classical solution $\hat{u} = u_{m,r,s,n,l,q}$ of (5.4) satisfying

$$\begin{aligned} \hat{u}(t,x) &= 0 && \text{on } \Sigma, \\ b_m(\hat{u}(0,x)) &= b_m(u_{0,q}(x)) && \text{on } \Omega. \end{aligned}$$

In what follows, we denote by $\hat{u}$ the function $u_{m,r,s,n,l,q}$. In order to study the convergence of the sequence $\hat{u}$ we shall need some uniform estimates in suitable functional spaces.

*A priori estimates.* By the maximum principle

$$|\hat{u}(t,x)| \leq M_1 \quad \forall (t,x) \in Q, \tag{5.7}$$

where $M_1$ is a positive constant independent of $m$, $r$, $s$, $l$ and $q$. On the other hand, if we denote by $\hat{v} := b_m(\hat{u})_t$ and we differentiate equation (5.4) with respect to $t$, we obtain that

(5.8)
$$\hat{v}_t = \operatorname{div}\left\{ \left( \frac{\partial \phi_r^j}{\partial \xi_i} \left( \frac{\partial}{\partial x_i} [(b_m^{-1})'(b_m(\hat{u}))\hat{v}] - k_s'(b_m(\hat{u}))e_i\hat{v} \right) \right)_{j=1,\dots,N} \right\} - g_n'(x,\hat{u})(b_m^{-1})'(b_m(\hat{u}))\hat{v} + \frac{\partial f_l}{\partial t}.$$

For any $\eta > 0$, we define the function $\mathcal{H}_\eta$ approximating the absolute value function in the following way: we first introduce

$$h_\eta(\sigma) = \begin{cases} \dfrac{2}{\eta}\left(1 - \dfrac{|\sigma|}{\eta}\right) & |\sigma| < \eta \\ 0 & |\sigma| \geq \eta, \end{cases}$$

and finally we define

$$H_\eta(\sigma) = \int_0^\sigma h_\eta(\tau)d\tau \text{ and } \boldsymbol{H}_\eta(\sigma) = \int_0^\sigma H_\eta(\tau)d\tau.$$

It is clear that

$$\begin{aligned} &h_\eta \geq 0, && \lim_{\eta \to 0} \sigma h_\eta(\sigma) = 0, \\ &|H_\eta| \leq 1, && \lim_{\eta \to 0} H_\eta(\sigma) = \operatorname{sgn}^0(\sigma) \end{aligned}$$

and

$$\lim_{\eta \to 0} \boldsymbol{H}_\eta(\sigma) = |\sigma|.$$

Multiplying equation (5.8) by $H_\eta(\hat{v})$, and integrating on $Q_t = ]0,t[\times\Omega$, we get

$$\begin{aligned}\int_\Omega \boldsymbol{H}_\eta(\hat{v}(t,x))\,dx - \int_\Omega \boldsymbol{H}_\eta(\hat{v}(0,x))\,dx \\ \leq -\int_{Q_t} (b_m^{-1})''(b_m(\hat{u}))\hat{v}h_\eta(\hat{v})\frac{\partial \phi_r^j}{\partial \xi_i}\frac{\partial b_m(\hat{u})}{\partial x_i}\frac{\partial \hat{v}}{\partial x_j}\,dx\,ds \\ + \int_{Q_t} \hat{v}h_\eta(\hat{v})k_s'(b_m(\hat{u}))\frac{\partial}{\partial \xi_i}\phi_r^j e_i \frac{\partial \hat{v}}{\partial x_j}\,dx\,ds \\ - \int_{Q_t} g_n'(x,\hat{u})(b_m^{-1})'(b_m(\hat{u}))\hat{v}H_\eta(\hat{v})\,dx\,ds \\ + \int_{Q_t} \frac{\partial f_l}{\partial t} H_\eta(\hat{v})\,dx\,ds,\end{aligned}$$

since $\hat{u}(s,x) = 0$ on $[0,T]\times\partial\Omega$, and then $\hat{v}(s,x) = 0$ on $[0,T]\times\partial\Omega$. Passing to the limit when $\eta \to 0$, we obtain the inequality

$$\begin{aligned}\int_\Omega |\hat{v}(t,x)|\,dx \leq \int_\Omega |\hat{v}(0,x)|\,dx + \int_{Q_t} \frac{\partial f_l}{\partial t}(s,x)\operatorname{sgn}^0(\hat{v})\,dx\,ds \\ - \int_{Q_t} g_n'(x,\hat{u})(b_m^{-1})'(b_m(\hat{u}))|\hat{v}|\,dx\,ds\end{aligned}$$

from the properties of $h_\eta$ and the monotonicity of the vectorial function $\phi_r$. Now, by (2.3), we arrive

$$-\int_{Q_t} g_n'(x,\hat{u})(b_m^{-1})'(b_m(\hat{u}))|\hat{v}| \leq C^* \int_{Q_t} |\hat{v}| b_m'(\hat{u})(b_m^{-1})'(b_m(\hat{u})).$$

Since $(b_m^{-1})'(b_m(s)) = 1/b_m'(s)$ wherever $b_m'(s) \neq 0$, we have that the last integral is equal to $\int_{Q_t - \{(s,x):b_m'(\hat{u}(s,x))=0\}} |\hat{v}|\,dx\,ds$. Taking this into account, one verifies that

$$\begin{aligned}\int_\Omega |\hat{v}(t,x)|\,dx \leq \int_\Omega |\hat{v}(0,x)|\,dx + \int_{Q_t} \frac{\partial f_l}{\partial t}(s,x)\operatorname{sgn}(\hat{v})\,dx\,ds \\ + C^* \int_{Q_t} |\hat{v}|\,dx\,ds.\end{aligned}$$

Using the equation satisfied by $\frac{\partial}{\partial t} b_m(\hat{u}(0,x))$ and the uniform boundedness of the data, we get

$$\begin{aligned}\int_\Omega \Big|\hat{v}(t,x))\Big|\,dx &\le \int_\Omega \left|\frac{\partial}{\partial t} b_m(\hat{u}(0,x))\right|\,dx + \int_{Q_t}\left|\frac{\partial f_l}{\partial t}\right|\,dx\,ds \\ &\quad + C^*\int_{Q_t} |\hat{v}|(s,x)\,dx\,ds \\ &\le \int_\Omega |\operatorname{div}\phi(\nabla u_{0,q} - k_s(b_m(u_{0,q}))e)|\,dx \\ &\quad + \int_\Omega |g_n(x,u_{0,q})|\,dx + \int_\Omega |f_l(0,x)|\,dx \\ &\quad + \int_{Q_t}\left|\frac{\partial f_l}{\partial t}\right|\,dx\,ds + C^*\int_{Q_t}|\hat{v}|(s,x)\,dx\,ds \\ &\le C_1 + C^*\int_{Q_t}|\hat{v}|(s,x)\,dx\,ds.\end{aligned}$$

Applying Gronwall's lemma, we obtain that

$$\int_\Omega |\hat{v}|\,dx \le e^{C^* t} \text{ for any } t\in[0,T].$$

Thus

$$\int_\Omega \left|\frac{\partial b_m(\hat{u})}{\partial t}\right|\,dx \le M_2 \text{ for any } t\in[0,T] \tag{5.9}$$

with $M_2 = e^{C^*T}$. From (5.7) and (5.3)

$$\int_\Omega \left|\frac{\partial}{\partial t}\hat{u}\right|\,dx \le M_3 \quad \forall t\in[0,T]. \tag{5.10}$$

Now, we shall show that there exists $M_4>0$, such that

$$\int_\Omega |\nabla\hat{u}|^p\,dx \le M_4 \tag{5.11}$$

uniformly in $t\in[0,T]$, $m$, $r$, $s$, $n$, $l$ and $q$. Firstly, we shall show that there exists an uniform positive constant $M'$ such that

$$\int_\Omega |\hat{\xi}|^p\,dx \le M' \quad \forall t\in[0,T] \tag{5.12}$$

where $\hat{\xi} := \nabla\hat{u} - k_s(b_m(\hat{u}))\boldsymbol{e}$. To do that, we multiply (5.4) by $\hat{u}$ and we integrate on $\Omega$: The

$$\int_\Omega \hat{u} b_m(\hat{u})_t + \int_\Omega \phi_r(\hat{\xi})\cdot\nabla\hat{u}\,dx + \int_\Omega g_n(x,\hat{u})\hat{u}\,dx = \int_\Omega f_l\hat{u}\,dx.$$

In what following it will appear several positive constants denoted by $C_i$, $i = 2, 3, 4, \dots$ which are independent on $t$ and the parameters $m$, $r$, $s$, $n$, $l$ and $q$. They will dependent on the exponent $p$, the measure of $\Omega$ and the above estimates. Some of them are also function of some positive parameters $\varepsilon$ and $\delta$ we shall introduce later. We shall only indicate the $\varepsilon$ and $\delta$ dependence.

By estimates (5.7) and (5.9), the assumption (2.3) on $g$, (5.1) on $f$ and the properties of $g_n$ and $f_l$, there exists a positive constant $C_2$ uniform in $m$, $r$, $s$, $n$, $l$ and $q$, such that

$$\int_\Omega \phi_r(\hat{\xi}) \cdot \hat{\xi}\, dx + \int_\Omega \phi_r(\hat{\xi}) k_s(b_m(\hat{u}))e\, dx \leq C_2$$

for any $t$. By Young's inequality, we have

$$\int_\Omega \phi_r(\hat{\xi}) \cdot \hat{\xi}\, dx \leq C_2 + \frac{\varepsilon^{p'}}{p'} \int_\Omega |\phi_r(\hat{\xi})|^{p'}\, dx + \frac{1}{\varepsilon^p p} \int_\Omega |k_s(b_m(\hat{u}))e|^p\, dx$$

where $\varepsilon$ is an arbitrary positive real number. The last integral is uniformly bounded in view of (5.7) and the assumptions on the sequences $\{k_s\}$ and $\{b_m\}$. By the properties of $\phi_r$, the first integral of the right hand side is bounded by $\int_\Omega |\hat{\xi}|^p\, dx$, for any $t$ in $[0, T]$. Hence,

$$\int_\Omega \phi_r(\hat{\xi}) \cdot \hat{\xi}\, dx \leq C_3(\varepsilon) + \frac{\varepsilon^{p'}}{p'} \int_\Omega |\hat{\xi}|^p\, dx \tag{5.13}$$

for some positive constant $C_3 = C_3(\varepsilon)$. Besides, from the properties of $\phi_r$, we have that

$$\alpha \int_\Omega \frac{|\hat{\xi}|^2}{(1 + |\hat{\xi}|)^{2-p}}\, dx \leq \int_\Omega \phi_r(\hat{\xi}) \cdot \hat{\xi}\, dx. \tag{5.14}$$

On the other hand, applying Young's inequality to $\int_\Omega \frac{|\hat{\xi}|^p}{(1+|\hat{\xi}|)^{2-p}} (1 + |\hat{\xi}|)^{2-p}\, dx$ with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$ we get

$$\int_\Omega |\hat{\xi}|^p\, dx \leq \frac{p}{2\delta^{2/p}} \int_\Omega \frac{|\hat{\xi}|^2}{(1 + |\hat{\xi}|)^{2-p}}\, dx + \delta^{2/(2-p)} \frac{2-p}{2} \int_\Omega (1 + |\hat{\xi}|)^p\, dx$$

for any $\delta > 0$. Hence

$$(1 - C_4 \delta^{2/(2-p)}) \int_\Omega |\hat{\xi}|^p\, dx \leq C_5(\delta) + \frac{p}{2\delta^{2/p}} \int_\Omega \frac{|\hat{\xi}|^2}{(1 + |\hat{\xi}|)^{2-p}}\, dx. \tag{5.15}$$

Using the estimates (5.13) and (5.14) into (5.15), we obtain the inequality

$$\alpha(1 - C_4\delta^{2/(2-p)}) \int_\Omega |\hat{\xi}|^p \, dx \le C_5 + \frac{p}{2\delta^{2/p}} \left\{ C_3 + \frac{\varepsilon^{p'}}{p'} \int_\Omega |\hat{\xi}|^p \, dx \right\},$$

which implies

$$\alpha \left( 1 - C_4\delta^{2/(2-p)} - C_6 \frac{\varepsilon^{p'}}{\delta^{2/p}} \right) \int_\Omega |\hat{\xi}|^p \, dx \le C_7(\varepsilon, \delta)$$

for some positive constant $C_7$ depending on $\varepsilon$, $\delta$. To verify the estimate (5.12), it is enough now to choose $0 < \delta \ll 1$ and $\varepsilon \ll 1$ such that $\varepsilon^{p'} \ll \delta^{2/p}$ and $1 - C_4\delta^{2/(2-p)} - C_6\frac{\varepsilon^{p'}}{\delta^{2/p}} > 0$. Now (5.12) implies (5.11) from the uniform boundedness of $\int_\Omega |k_s(b_m(\hat{u}))|^p \, dx$.

Finally, multiplying the relation (5.4) by $v \in X'$ and integrating on $Q$, we obtain, using the Hölder's inequality that

$$\begin{aligned} \left| \int_Q v \frac{\partial b_m(\hat{u})}{\partial t} \right| \le & \left[ \int_Q |\phi_r(\hat{\xi})|^{p'} \, dx\, dt \right]^{1/p'} \left[ \int_Q |\nabla v|^p \, dx\, dt \right]^{1/p} \\ & + \|v\|_{L^\infty(Q)} \int_Q |g_n(x, \hat{u})| \, dx\, dt \\ & + \left[ \int_Q |f_l|^{p'} \, dx\, dt \right]^{1/p'} \left[ \int_Q |v|^p \, dx\, dt \right]^{1/p}. \end{aligned}$$

The properties (5.5) and (5.6) of $\phi_r$, the assumptions (2.2) on $g$ and (5.1) on $f$ and the properties on $g_n$ and $f_l$ lead to estimate

$$\left| \int_Q v \frac{\partial b_m(\hat{u})}{\partial t} \right| \le M_5 \|v\|_{L^p(0,T;W_0^{1,p}(\Omega)) cap L^\infty(Q)}$$

for some positive constant $M_5$ independent on $m$, $r$, $s$, $n$, $l$ and $q$ where we used estimates (5.7) and (5.12). In this way, we obtain the following uniform estimate in $L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$

$$\left\| \frac{\partial b_m(\hat{u})}{\partial t} \right\|_{L^{p'}(o,T;W^{-1,p'}(\Omega)) + L^1(Q)} \le M_5. \tag{5.16}$$

*Passing to the limit.* By the estimates (5.7), (5.9), (5.10), (5.11) and (5.16) we can find a bounded BV solution of problem (1.1), (1.2) and (1.3) as limit of some subsequence of $\{\hat{u}\} := \{u_{m,r,s,n,l,q}\}$ (which we will denote again by $\{\hat{u}\}$). Moreover, this solution belongs to $C([0,T],L^1(\Omega))$. Indeed by the estimates (5.7), (5.10) and (5.11) and Corollary 4 of Simon **[S87]**, there exists a subsequence of $\{\hat{u}\}$ (called again $\{\hat{u}\}$) and a function $u \in C([0,T],L^1(\Omega))$ such that $\hat{u} \to u$ in $C([0,T],L^1(\Omega))$. In particular,

$$\hat{u} \to u \text{ in } L^1(Q)$$

and

$$\hat{u} \to u \text{ a.e. in } Q$$

(except, perhaps, for a subsequence). By (5.7)

$$\hat{u} \stackrel{*}{\rightharpoonup} u \text{ in } L^\infty(Q) \text{ weakly*}$$

and

$$\hat{u} \rightharpoonup u \text{ in } L^p(0,T;W_0^{1,p}(\Omega))$$

from (5.11). By (5.7) and the assumption on $b_m$ we can deduce the weak* convergence $b_m(\hat{u}) \stackrel{*}{\rightharpoonup} \beta$ in $L^\infty(Q)$ for some $\beta \in L^\infty(Q)$. The fact that $\hat{u}$ converges to $u$ almost everywhere of $Q$ and the properties of $b_m$ and $b$ imply that $b_m(\hat{u}) \to b(u)$ a.e. point of $Q$. By Lebesgue's Theorem there is strong convergence of $b_m(\hat{u})$ to $b(u)$ in $L^\sigma(Q)$ $(1 \le \sigma < \infty)$. Analogously, $b_m(u_{0,q})$ goes to $b(u_0)$ strongly in $L^\sigma(\Omega)$ $(1 \le \sigma < \infty)$. Now, by (5.9) and as $b_m(\hat{u}) \to b(u)$ in $L^1(Q)$, we have that $b(u) \in BV_t(Q)$. Finally, $\frac{\partial b_m(\hat{u})}{\partial t} \to \frac{\partial b(u)}{\partial t}$ in the sense of distributions. Moreover, $\frac{\partial b_m(\hat{u})}{\partial t}$ converges to $\frac{\partial b(u)}{\partial t}$ weakly in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + \mathrm{L}^1(Q)$ from (5.16). By usual argument, we obtain that $g_n(x,\hat{u})$ converges to $g(x,u)$ in $L^1(Q)$. Since $\hat{u}$ is bounded in $L^\infty(0,T;W_0^{1,p}(\Omega))$, then $\phi_r(\nabla\hat{u} - k_s(b_m(\hat{u}))e)$ is also bounded in $L^\infty(0,T;(L^{p'}(\Omega))^N)$ and thus there exists a subsequence of $\{\hat{u}\}$ (again called $\{\hat{u}\}$ such that $\phi_r(\nabla\hat{u} - k_s(b_m(\hat{u}))e)$ converges to $Y$ weakly* in $L^\infty(0,T;(L^{p'}(\Omega))^N)$. Multiplying the equation (5.4) by a test function $v \in L^p(0,T;W_0^{1,p}(\Omega)) \cap \mathrm{L}^\infty(Q)$ and integrating on $Q$, we obtain that

$$\text{(5.17)} \quad \int_Q v\frac{\partial b_m(\hat{u})}{\partial t} + \int_Q \phi_r(\nabla\hat{u} - k_s(b_m(\hat{u}))e)\cdot\nabla v + \int_Q g_n(x,\hat{u})v = \int_Q f_l v.$$

Let us see that $u$ verifies (2.9). To do that, we pass to the limit in the variables in (5.17) when $m$, $r$, $s$, $n$, $l$ and $q \to \infty$. By the above convergences, we arrive to

$$\text{(5.18)} \qquad \left\langle \frac{\partial b(u)}{\partial t}, v \right\rangle_{X,X'} + \int_Q Y\cdot\nabla v + \int_Q g(x,u)v = \int_Q fv.$$

for all $v \in L^p(0,T;W_0^{1,p}(\Omega)) \cap \mathrm{L}^\infty(Q)$. We have to prove now that

$$Y \equiv \phi(\nabla u - k(b(u))e) \tag{5.19}$$

which is not completely obvious due to the nonlinear character of the differential operator. We shall prove this by using Minty's type argument (see also [**DT94**]). We shall show that

$$\int_\Omega [Y - \phi(\nabla\chi - k(b(u))e)] \cdot [\nabla u - \nabla\chi]\,dx \geq 0, \text{ for any } \chi \in W_0^{1,p}. \tag{5.20}$$

Then, we can obtain (5.19) by taking $\xi \in W_0^{1,p}(\Omega)$ arbitrary and the function $\chi = u - \lambda\xi$ with $\lambda > 0$ ($\lambda < 0$). To prove (5.20), we take $0 \leq \varphi \in c_c^\infty 0, T$ and for any $\chi \in W_0^{1,p}(\Omega)$ we use the decomposition

$$\int_Q [\phi_r(\nabla\hat{u} - k_s(b_m(\hat{u}))e) - \phi(\nabla\chi - k(b(u))e)] \cdot \nabla(u-\chi)\varphi(t) = I_1 + I_2 + I_3 + I_4 \tag{5.21}$$

where

$$I_1 = \int_Q \phi_r(\nabla\hat{u} - k_s(b_m(\hat{u}))e) \cdot \nabla(u-\hat{u})\varphi(t)$$

$$I_2 = \int_Q [\phi_r(\nabla\hat{u} - k_s(b_m(\hat{u}))e) - \phi_r(\nabla\chi - k_s(b_m(\hat{u}))e)] \cdot \nabla(\hat{u}-\chi)\varphi(t)$$

$$I_3 = \int_Q \phi_r(\nabla\chi - k_s(b_m(\hat{u}))e) \cdot \nabla(\hat{u}-u)\varphi(t)$$

and

$$I_4 = \int_Q [\phi_r(\nabla\chi - k_s(b_m(\hat{u}))e) - \phi(\nabla\chi - k(b(u))e)] \cdot \nabla(u-\chi)\varphi(t).$$

Due to monotonicity of $\phi_r$, the integral $I_2$ is non negative. On the other hand

$$\lim \int_Q |\phi_r(\nabla\chi - k_s(b_m(\hat{u}))e) - \phi(\nabla\chi - k(b(u))e)|^{p'} = 0 \tag{5.22}$$

from the properties of $\phi_r, k_s, b_m, \hat{u}$. Thus $\lim I_3 = 0 = \lim I_4$. Finally, multiplying the equation (5.4) by $u\varphi(t)$ and $\hat{u}\varphi(t)$, integrating and subtracting, we obtain that

$$I_1 = \int_Q \phi_r(\nabla \hat{u} - k_s(b_m(\hat{u}))e) \cdot \nabla(u - \hat{u})\varphi(t)$$

$$= -\int_Q \frac{\partial b_m(\hat{u})}{\partial t} u\varphi(t) \tag{5.23}$$

$$+ \int_Q \frac{\partial b_m(\hat{u})}{\partial t} \hat{u}\varphi(t) \tag{5.24}$$

$$- \int_Q g_n(x, \hat{u})(u - \hat{u})\varphi(t) \tag{5.25}$$

$$+ \int_Q f_l(u - \hat{u})\varphi(t). \tag{5.26}$$

The integrals (5.25) and (5.26) converge to zero when $m$, $r$, $s$, $n$, $l$, $q \to \infty$. The weak convergence of $b_m(\hat{u})_t$ to $b(u)_t$ in $X$ and the fact that $u\varphi(t) \in X'$, imply that the integral (5.23) (i.e. $-\langle \frac{\partial b_m(\hat{u})}{\partial t}, u\varphi(t)\rangle_{X,X'}$) converges to

$$-\left\langle \frac{\partial b(u)}{\partial t}, u\varphi(t) \right\rangle_{X,X'}. \tag{5.27}$$

We shall also show that the integral (5.24) converges to (5.27) as in [**DT94**]. We define

$$B_m(\eta) = \int_0^\eta (b_m(\eta) - b_m(s))\, ds \quad \forall \eta \in \mathbb{R}$$

and

$$z_{\hat{u}}(t) = \int_\Omega B_m(\hat{u}(t, x))\, dx.$$

It is easy to see that $B_m(\hat{u})$ is bounded in $Q$ and thus

$$\|z_{\hat{u}}(t)\|_{L^1(0,T)} \text{ is uniformely bounded.}$$

As in Lemma 2 of Bamberger [**Ba77**] we get that

$$\int_\Omega \hat{u}(t) \frac{\partial b_m(\hat{u})(t)}{\partial t}\, dx = \left\langle \frac{\partial b_m(\hat{u})(t)}{\partial t}, \hat{u}(t) \right\rangle = \frac{dz_{\hat{u}}}{dt}(t)$$

a.e. $t \in ]0, T[$, in the sense of $\mathcal{D}'(0, T)$. Now, thanks to the convergence of $\hat{u}$ and $b_m(\hat{u})$ and the boundedness of $z_{\hat{u}}$ in $L^1(0, T)$ we obtain that

$$z_{\hat{u}} \to z_u \text{ in } L^1(0, T) \text{ and a.e. in } ]0, T[. \tag{5.28}$$

Since $u \in X'$ and $b(u)_t \in X$, we arrive to

$$\langle b(u)(t)_t, u(t)\rangle = \frac{dz_u}{dt}(t) \tag{5.29}$$

for a.e. $t \in ]0,T[$, in $\mathcal{D}'(0,T)$, obtaining

$$\begin{aligned}\int_Q b_m(\hat{u})_t \hat{u}\varphi(t)\,dx\,dt &= \int_0^T \varphi(t)\left(\int_\Omega \frac{\partial b_m(\hat{u})}{\partial t}\hat{u}\,dx\right)dt\\ &= \left\langle \varphi(t)\frac{dz_{\hat{u}}}{dt}(t)\right\rangle \quad (\text{in } \mathcal{D}'(0,T))\\ &= -\int_0^T z_{\hat{u}}(t)\frac{d\varphi}{dt}(t).\end{aligned}$$

Passing to the limit,

$$\lim \int_Q \frac{\partial b_m(\hat{u})}{\partial t}\hat{u}\varphi(t) = -\int_0^T z_u(t)\frac{d\varphi}{dt}(t)$$

by (5.28) because $\varphi \in c_c^1 0, T$. Thus

$$\begin{aligned}\lim \int_Q \frac{\partial b_m(\hat{u})}{\partial t}\hat{u}\varphi(t) &= -\int_0^T z_u(t)\frac{d\varphi}{dt}(t)\\ &= \int_0^T \left\langle \frac{\partial b(u)}{\partial t}, u\varphi(t)\right\rangle_{X,X'} dt\\ &= \left\langle \frac{\partial b(u)}{\partial t}, u\varphi(t)\right\rangle_{X,X'}\end{aligned}$$

and so $\lim I_1 = 0$.

Summarizing: we have proved that the limits of integrals $I_1$, $I_2$, $I_3$, $I_4$ are non negative and thus (5.19) holds. Then, $u$ satisfies the equation (2.9). By standard arguments we can see that $u$ verifies also (2.8).

*Case $p \geq 2$:* As in the case $1 < p < 2$, we begin by defining a family of regular problems, we find suitable a priori estimates and finally we obtain $u$ as the limit of the regular solutions associated to the family of regular problems. The family of regularized problems can be defined now by

$$\frac{\partial b_m(u)}{\partial t} - \operatorname{div}\phi(\nabla u - k_s(b_m(u))e)$$

$$-\epsilon\Delta u + g_n(x,u) = f_l(t,x) \quad \text{in } Q, \tag{5.30}$$

$$u(t,x) = 0 \quad \text{on } \Sigma, \tag{5.31}$$

$$b_m(u(0,x)) = b(u_{0,q}(x)) \quad \text{in } \Omega \tag{5.32}$$

with $b_m, k_s, g_n, f_l$ and $u_{0,q}$ as in the case $1 < p < 2$ and $\epsilon > 0$. The existence of a classical solution is again a well-known result (see [**LSU68**, Chapter V]). The rest of details follows the same arguments. ■

The above theorem proves the existence of BV solution of problem (1.1), (1.2) and (1.3). Nevertheless our uniqueness results on BV solutions we will need some additional assumptions. The following corollary gives an answer in this sense.

**Corollary 7.** *If in addition to the assumptions of Theorem 2, we suppose that*

$$k \circ b \text{ is locally Lipschitz if } 1 < p \leq 2 \tag{5.33}$$

*or*

$$k \circ b(\sigma) = \lambda\sigma + \nu \text{ for some } \lambda, \nu \in \mathbb{R} \text{ if } p > 2, \tag{5.34}$$

*then, there exists a BV solution u of problem (1.1), (1.2), (1.3), such that*

$$\frac{\partial u}{\partial t} \in L^2(Q). \tag{5.35}$$

*Proof:* We use the same technique that in the proof of Theorem 2. Due to that, we shall made mention only to the new arguments. Let $\hat{u}$ be the solution of the regularized problems. As before, we obtain the estimates (5.7), (5.11), (5.9). Now, we shall find an $L^2(Q)$ uniform estimate on $\frac{\partial \hat{u}}{\partial t}$. In the case $1 < p < 2$, we multiply the equation (5.4) by $\frac{\partial \hat{u}}{\partial t}$ and integrate on $Q$. Then

$$\int_Q b_m(\hat{u})_t \hat{u}_t = \int_Q f_l \hat{u}_t - \int_Q \frac{\partial}{\partial t} G_n(x, \hat{u}) + \int_Q \operatorname{div} \phi(\hat{\xi}) \hat{u}_t,$$

where $G_n(x, \cdot)$ is such that $\frac{\partial}{\partial s} G_n(\cdot, s) = g_n(\cdot, s)$. If we denote by $\Phi_r$ a primitive of $\phi_r$, we obtain the equality

$$\begin{aligned}
\int_Q b_m(\hat{u})_t \hat{u}_t = & \int_\Omega f_l(T, x)\hat{u}(T, x) - \int_\Omega f_l(0, x)\hat{u}(0, x)\, dx - \int_Q \frac{\partial f}{\partial t} \hat{u} \\
& - \int_\Omega G_n(x, \hat{u}(T, x)) + \int_\Omega G(x, \hat{u}(0, x))\, dx \\
& + \int_\Omega \Phi_r(\nabla \hat{u}(0, x) - k_s(b_m(\hat{u}(0, x))e) \\
& - \int_\Omega \Phi_r(\nabla \hat{u}(T, x) - k_s(b_m(\hat{u}(T, x))e) \\
& - \int_Q e \cdot \phi_r(\hat{\xi}) \frac{\partial}{\partial t}[k_s \circ b_m(\hat{u})].
\end{aligned}$$

By the estimates (5.7), (5.2) and (5.1) and since $\Phi_r$ is non negative, we have

$$\int_Q b_m(\hat{u})_t \hat{u}_t \leq C_1 + \int_Q |e \cdot \phi_r(\hat{\xi})| \left|\frac{\partial}{\partial t}[k_s \circ b_m(\hat{u})]\right|,$$

with $C_1$ a constant independent on $m, r, s, n, l, q$. By Young's inequality,

$$\int_Q b_m(\hat{u})_t \hat{u}_t \leq C_1 + \frac{1}{\varepsilon^{p'} p'} \int_Q |\phi_r(\hat{\xi})|^{p'} + \frac{\varepsilon^p}{p} \int_Q \left|\frac{\partial}{\partial t}[k_s \circ b_m(\hat{u})]\right|^p, \tag{5.36}$$

for all $\varepsilon > 0$. Now, since $\hat{u}$ is uniformly bounded (see (5.7)) and since we have assumed (5.33), then there exists a positive constant $L_{kob}$ such that for all $s$, $m \in \mathbb{N}$ $L_{kob} \geq \mathrm{lip}(k_s \circ b_m, [-M_1, M_1])$ (:=the Lipschitz constant of $k \circ b$ in the interval $[-M_1, M_1]$). Thus,

$$\int_Q \left|\frac{\partial}{\partial t}[k_s \circ b_m(\hat{u})]\right| \leq L_{kob} \int_Q \left|\frac{\partial \hat{u}}{\partial t}\right|.$$

On the other hand, as $b$ verifies (3.5), we obtain

$$L \int_Q |\hat{u}_t|^2 \leq \int_Q b(\hat{u})_t u_t$$

for some positive constant $L$ independent on $m, r, s, n, l, q$. Considering the above inequalities and estimate (5.11), from (5.36) we arrive to

$$\frac{\lambda_1 - L_1 L_2 \lambda_2}{L_1} \int_Q |\hat{u}_t|^2 \leq C_1 + L_{kob}^p \frac{\varepsilon}{p} \int_Q |\hat{u}_t|^p$$

for some positive constant $C_2$. Finally, since $1 < p < 2$ and $Q$ is bounded, applying the Hölder's inequality we get

$$\int_Q \left|\frac{\partial \hat{u}}{\partial t}\right|^2 \leq C_3 \tag{5.37}$$

with $C_3$ a positive constant independent on $m, r, s, n, l, q$. This new estimate jointly with the estimates given in the Theorem 2 allows us to show that the BV solution obtained as limit of the sequence $\{\hat{u}\}$ verifies (5.35).

In the case $p \geq 2$, we can suppose, without lost of generality, that $e = e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$. Multiplying the equation (5.30) by $\hat{u}_t e^{-\lambda x_1}$ and applying (5.34), we obtain

$$\begin{aligned}\int_Q |\hat{u}_t|^2 e^{-\lambda x_1} &\leq \int_Q b_m(\hat{u})_t u_t e^{-\lambda x_1} \\ &\leq C_1 + \left|\int_Q \frac{\partial}{\partial t} \Phi(\nabla \hat{u} - k_s(b_m(\hat{u}))e_1) e^{-\lambda x_1}\right|\end{aligned}$$

with $\Phi(\xi) = \frac{1}{p}|\xi|^p$, $\xi \in \mathbb{R}^N$ and $C_1$ a positive constant independent on $m, s, n, l, q, \epsilon$ thanks to the previous estimates on $\hat{u}$ and $|\nabla \hat{u}|$. Finally, as in the case $1 < p < 2$, we conclude that $u$ is a bounded BV satisfying (5.35). ■

**Remark 9.** The bounded BV solution $u$ obtained in Theorem 2 belongs to $W^{1,1}(Q)$. Then, by Lemma 1, the Hausdorff $N$-dimensional measure of the set of jumping points of $u$ is zero. Then, by Corollary 3, this solution is unique in this class of solutions. An other way to obtain the above conclusion is by applying Corollary 6 with $\Psi_1$ and $\Psi_2$ the identity, since any pair of solutions $u_1, u_2$ obtained as in Theorem 2 are in the $W^{1,1}(Q)$ space.

## References

[AL83] H. W. ALT AND S. LUCKHAUS, Quasilinear Elliptic-Parabolic Differential Equations, *Math. Z.* **183** (1983), 311–341.

[Ba77] A. BAMBERGER, Etude d'une équation doublement non linéaire, *J. Funct. Anal.* **24** (1977), 148–155.

[Be81] PH. BENILAN, "*Evolution equations and accretive operators*," Lecture Notes, Univ. of Kentucky, 1981.

[BeGa95] PH. BENILAN AND R. GARIEPY, Strong solutions in $L^1$ of degenerate parabolic equations, *Journal Differential Equations* **119** (1995), 473–502.

[C86] J. CARRILLO, Unicité des solutions du type Kruskov pour des problèms elliptiques avec des terms de transport non linéaires, *C. R. Acad. Sci. Paris Sér. I Math.* **303(5)** (1986), 189–192.

[DLi89] J. I. DÍAZ AND A. LIÑÁN, Tiempo de descarga en oleoductos o gaseoductos largos: Modelización y estudio de una ecuación parabólica doblemente no lineal, in "*Actas de la Reunión Matemática en Honor a A. Dou.*," (J. I. Díaz and J. M. Vegas, eds.), Univ. Complutense, 1989, pp. 95–120.

[DT94] J. I. DÍAZ AND F. DE THELIN, On a nonlinear parabolic problem arising in some models related to turbulent flows, *SIAM J. Math. Anal.* **25(4)** (1994), 1085–111.

[DPa93] J. I. DÍAZ AND J. F. PADIAL, Soluciones en el espacio BV de un problema parabólico no lineal asociado a modelos de flujos turbulentos, in "*III Congreso de Matemática Aplicada, XIII C.E.D.Y.A.*," (A. C. Casal et al., eds.), Universidad Politécnica de Madrid, 1993, pp. 304–309.

[EvG92] L. C. EVANS AND R. F. GARIEPY, "*Measure Theory and Fine Properties of Functions*," CRC Press, Inc., 1992.

[F69] H. FEDERER, "*Geometric Measure Theory*," Springer-Verlag, Berlin, Heidelberg, New York, 1969.

[GM92a] G. GAGNEUX AND M. MADAUNE-TORT, Sur la dimension de Hausdorff de l'onde de choc pour des models non linéaires de la diffusion-convection, *Publications de l'URA* **1204**, *Analyse Num.* (1993).

[GM92b] G. GAGNEUX AND M. MADAUNE-TORT, Sur la question de l'unicité pour des inéquations des milieux poreux, *C. R. Acad. Sci. Paris Sér. I Math.* **314** (1992), 605–608.

[GM94] G. GAGNEUX AND M. MADAUNE-TORT, Unicité des solutions faibles d'équations de diffusion-convection, *C. R. Acad. Sci. Paris Sér. I Math.* **318** (1994), 919–924.

[GM95] G. GAGNEUX AND M. MADAUNE-TORT, "*Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière*," Collection "Mathématiques et applications", Springer-Verlag **22**, 1995.

[J89] Y. JINGXUE, Some properties of functions in $BV_x$ and their applications to the uniqueness of solutions for degenerate quasilinear parabolic equations, *Northeast. Math. J.* **5(4)** (1989), 395–422.

[J90a] Y. JINGXUE, On a class of quasilinear parabolic equations of second order with double–degeneracy, *J. Partial Differential Equations* **3(4)** (1990), 49–64.

[J90b] Y. JINGXUE, On the uniqueness and stability of BV solutions for nonlinear diffusion equations, *Comm. Partial Differential Equations* **15(12)** (1990), 1671–1683.

[J92] Y. JINGXUE, A note on the uniqueness for double degenerate nonlinear parabolic equations, *J. Partial Differential Equations* **5(2)** (1992), 94–96.

[LSU68] O. A. LADYZENSKAJA, V. A. SOLONNIKOV AND N. N. URALCEVA, "*Linear and Quasilinear Equations of Parabolic Type*," Translations of the American Mathematical Society Providence, Rhode Island, 1968.

[Pa95] J. F. PADIAL, EDP's no lineales originadas en plasmas de fusión y filtración en medios porosos, Tesis de la Universidad Complutense de Madrid, Julio 1995.

[P95] A. PLOUVIER, Sur une classe de problèmes d'évolution quasi linéaires dégénérés, *Rev. Mat. Univ. Complut. Madrid* **8(1)** (1995).

[PG96] A. PLOUVIER-DEBAIGT AND G. GAGNEUX, Unicité de la solution faible d'équations des milieux poreus, via un concept de solution renormalisé, *C. R. Acad. Sci. Paris Sér. I Math.* **322** (1996), (to appear).

[S87] S. SIMON, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.(4)* **CXLVI** (1987), 65–96.

[U96] P. URRUTY, Un concept de solution renormalisée pour l'étude d'une loi générale de conservation (to appear).

[V69] R. E. VOLKER, Nonlinear flow in porous media by finite elements, *J. Hydraulics Div. Proc. Amer. Soc. Civil. Eng* **95** (1969), 2093–2114.

[Vo67] A. I. VOL'PERT, The spaces BV and quasilinear equations, *Mat. Sb.* **73(115), no. 2** (1967).

[VoHu69] A. I. VOL'PERT AND S. I. HUDJAEV, Cauchy's problem for degenerate second order quasilinear parabolic equations, *Mat. Sb.* **78(120), no. 3** (1969).

[VoHu85] A. I. VOL'PERT AND S. I. HUDJAEV, "*Analysis in classes of discontinuous functions and equations of mathematical physics*," Martinus Nijhoff Publishers, Dordrecht, 1985.

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Rebut el 23 de Juliol de 1996