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Some results about the approximate controllability property for quasilinear diffusion equations

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Abstract. We study the approximate controllability property for $y_t - \Delta\varphi(y) = u\chi_\omega$, on $\Omega\times(0,T)$, where Ω is a bounded open set of $\mathbb{R}^N$ and $\omega \subset \Omega$. First, we show some negative results for the case $\varphi(s) = |s|^{m-1}s$, $0 < m < 1$, by means of an obstruction phenomenon. In a second part, we obtain a positive answer on the space $H^{-1-\gamma}(\Omega)$, for any $\gamma > 0$, for a class of functions φ essentially linear at infinity, by using a higher order vanishing viscosity argument.

Quelques résultats sur la propriété de la contrôlabilité approchée pour les équations de diffusion quasi-linéaires

Résumé. *On étudie la propriété de la contrôlabilité approchée pour* $y_t(x,t) - \Delta\varphi(y(x,t)) = u(x,t)\chi_\omega$, *où* $(x,t) \in \Omega \times (0,T)$ *et* $\omega \subset \Omega$. *D'abord, par un phénomène d'obstruction, on montre quelques résultats négatifs pour les fonctions* $\varphi(s) = |s|^{m-1}s$, $0 < m < 1$. *Finalement, on donne une réponse positive dans les espaces* $H^{-1-\gamma}(\Omega)$, $\gamma > 0$ *arbitraire, pour une famille de fonctions qui sont essentiellement linéaires à l'infini, en utilisant un argument de viscosité d'ordre supérieur.*

Version française abrégée

On étudie la contrôlabilité approchée de l'équation de diffusion quasi-linéaire

$$\mathcal{P}(u)\begin{cases} y_t(x,t) - \Delta\varphi(y(x,t)) = h(x,t) + u(x,t)\chi_\omega, & (x,t) \in Q := \Omega \times (0,T), \\ \varphi(y(x,t)) = 0, & (x,t) \in \Sigma := \partial\Omega \times (0,T), \\ y(x,0) = y_0(x), & x \in \Omega, \end{cases}$$

où Ω est un ouvert borné régulier de $\mathbb{R}^N$, ω un sous-ensemble ouvert de Ω, $h \in L^2(0,T : H^{-1}(\Omega))$ et $y_0 \in L^2(\Omega)$ sont des données, et $u \in L^2(\omega \times (0,T))$ le contrôle. La fonction φ est supposée continue, non décroissante et sous-linéaire à l'infini. L'existence et l'unicité d'une solution

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$y \in \mathcal{C}([0,T]:H^{-1}(\Omega))$ sont bien connues (*voir* [2]). Étant donnés $\gamma > 0$, $y_d \in H^{-1-\gamma}(\Omega)$, $T > 0$ et $\delta > 0$, on veut savoir s'il existe $u \in L^2(\omega \times (0,T))$ tel que la solution y de $\mathcal{P}(u)$ vérifie

$$\|y(T) - y_d\|_{H^{-1-\gamma}(\Omega)} \le \delta. \tag{1}$$

On commence par montrer un *phénomène d'obstruction* pour $\varphi(s) = |s|^{m-1}s$, $0 < m < 1$. Bien que la classe restante de fonctions sous-linéaires soit assez étroite, on montre la contrôlabilité approchée dans $H^{-1-\gamma}(\Omega)$ pour des fonctions φ qui sont essentiellement linéaires à l'infini. Cette classe de fonctions inclut celle qui correspond au problème de Stefan à deux phases.

THÉORÈME 1. – *Soit $y(\cdot\,;u) \in \mathcal{C}([0,T]:L^1(\Omega))$ solution de $\mathcal{P}(u)$, avec $\varphi(s) = |s|^{m-1}s$, $0 < m < 1$, $y_0 \in L^2(\Omega)$, $h \equiv 0$ et $u \in L^1(\omega \times (0,T))$. Alors, on peut choisir un état $y_d \in L^1(\Omega)$ tel que $\|y(T;u) - y_d\|_{L^1(\Omega)} > \delta$, quelque soit le contrôle $u \in L^1(\omega \times (0,T))$, avec $\delta > 0$ assez petit.*

La démonstration est liée au résultat suivant :

LEMME 1 (*voir* [7]). – *Soient $m \in (0,1)$, $R > 0$ et y et $\hat{y} \in \mathcal{C}([0,T]:L^1(B_R(x_0)))$ deux fonctions satisfaisant l'équation $y_t - \Delta(|y|^{m-1}y) = 0$ dans $\mathcal{D}'(B_{2R}(x_0) \times (0,T))$. Supposons $y \ge \hat{y}$. Alors, si $\alpha = 1/(1-m)$ et $\beta = 2/(1-m) - N$, pour tous t, $s \in [0,T]$, il existe $C = C(N,m)$ tel que*

$$\int_{B_R(x_0)} |y(t) - \hat{y}(t)| \le C\left[\int_{B_{2R}(x_0)} \left(|y(s) - \hat{y}(s)| + |t-s|^{\alpha} R^{-\beta}\right)\right]. \tag{2}$$

REMARQUE 1. – Grâce au Principe de Comparaison, on peut aussi montrer, dans le cas où $\varphi(s) = |s|^{m-1}s$, $0 < m < 1$, l'existence de U_- et U_+ telles que $U_-(x,t) \le y(x,t;u) \le U_+(x,t)$ pour tout $t \in [0,T]$, p.p. tout $x \in \Omega\backslash\omega$ et pour tout $u \in L^2(\omega \times (0,T))$. Si $m \in (1/2, 1)$ ces inégalités impliquent aussi une obstruction dans $H^{-2}(\Omega)$ et donc dans les espaces $X \subset H^{-2}(\Omega)$.

THÉORÈME 2. – *Soit φ continue non décroissante, $\varphi \in \mathcal{C}^1(\mathbb{R}\backslash[-M_1, M_1])$ pour une certaine constante $M_1 \ge 0$ et $\varphi(0) = 0$. Supposons qu'il existe k et C_1 telles que*

$$|\varphi'(s) - k| \le \frac{C_1}{|s|} \quad \forall |s| > M_1, \tag{3}$$

$$|\varphi(s) - ks| \le C_2 \quad \forall s \in \mathbb{R}. \tag{4}$$

Alors, si $\varphi'(s) \ge c > 0$ p.p. tout $s \in \mathbb{R}$ ou $h \in L^2(Q)$, on a (1).

Pour démontrer ce théorème, on passe à la limite, lorsque $\varepsilon \to 0$, sur les problèmes d'ordre supérieur

$$\mathcal{P}_\varepsilon(u)\begin{cases} y_t(x,t) + \varepsilon\Delta^2 y(x,t) - \Delta\varphi(y(x,t)) = h(x,t) + u(x,t)\chi_\omega, & (x,t) \in Q, \\ y(x,t) = \Delta y(x,t) = 0, & (x,t) \in \Sigma, \\ y(x,0) = y_0(x), & x \in \Omega. \end{cases}$$

THÉORÈME 3. – *On suppose φ vérifiant les hypothèses du Théorème 2. Alors, étant donnés $\gamma > 0$ et $y_d \in H^{-1-\gamma}(\Omega)$, il existe K, qui est indépendante de ε, et une suite de contrôles $v_\varepsilon \in L^2(\omega \times (0,T))$ vérifiant $\|v_\varepsilon\|_{L^2(\omega\times(0,T))} \le K$, $\forall \varepsilon > 0$, telles que les solutions $y(\cdot\,;v_\varepsilon)$ de $\mathcal{P}_\varepsilon(v_\varepsilon)$ vérifient $\|y(T;v_\varepsilon) - y_d\|_{H^{-1-\gamma}(\Omega)} < \delta$.*

1. Introduction

We study the approximate controllability of problem $\mathcal{P}(u)$, where Ω is a bounded open subset of $\mathbb{R}^N$, $T > 0$, ω is a nonempty open subset of Ω, $h \in L^2(0,T:H^{-1}(\Omega))$, and $y_0 \in L^2(\Omega)$

are prescribed data, and $v \in L^2(\omega \times (0,T))$ represents the searched output control. The function φ is assumed to be continuous, nondecreasing and *sublinear at infinity* (*i.e.* $|\varphi(s)| \leq C(1+|s|)$ for $|s| > M$). The existence and uniqueness of a solution $y \in \mathcal{C}([0,T] : H^{-1}(\Omega))$ are well-known (*see* [2]). Given $\gamma > 0$, for any desired state $y_d \in H^{-1-\gamma}(\Omega)$, any fixed time T, and any $\delta > 0$, we want to find a control $u \in L^2(\omega \times (0,T))$ such that (1) holds. We start by proving that (in contrast with the case of semilinear equations; *see* [6]) an *Obstruction Phenomenon* (implying the impossibility of the approximate controllability) arises in the case of φ being strictly sublinear ($\varphi(s) = |s|^{m-1}s$, $0 < m < 1$). Although the remaining range of sublinear functions is quite narrow, we shall show that the approximate controllability holds for functions φ which are *essentially linear at infinity*. This class of functions includes the one associated to the two-phase Stefan Problem. The result is obtained via the vanishing viscosity higher order problem $y_t + \varepsilon\Delta^2 y - \Delta\varphi(y) = h + u\chi_\omega$.

2. Obstruction phenomenon when the nonlinearity is strictly sublinear

THEOREM 1. – *Let $y(\cdot\,;u) \in \mathcal{C}([0,T] : L^1(\Omega))$ be a solution of $\mathcal{P}(u)$ with $\varphi(s) = |s|^{m-1}s$, $0 < m < 1$, $y_0 \in L^1(\Omega)$, $h \equiv 0$, and $u \in L^1(\omega \times (0,T))$. Then we can choose $y_d \in L^1(\Omega)$ such that $\|y(T;u) - y_d\|_{L^1(\Omega)} > \delta$ for any $u \in L^1(\omega \times (0,T))$ and any $\delta > 0$ small enough.*

The proof is based on the following lemma:

LEMMA 1 (*see* [7]). – *Let $m \in (0,1)$ and $R > 0$. Let y, $\hat{y} \in \mathcal{C}([0,T] : L^1(B_R(x_0)))$ satisfying $y_t - \Delta(|y|^{m-1}y) = 0$ in $\mathcal{D}'(B_{2R}(x_0) \times (0,T))$. Assume that $y \geq \hat{y}$. Then, for any t, $s \in [0,T]$, there exists $C = C(N,m)$ such that inequality* (2) *holds, where $\alpha = 1/(1-m)$ and $\beta = 2/(1-m) - N$.*

Proof of Theorem 1. – Let $x_0 \in \Omega\backslash\omega$ and $R > 0$ be such that $B_{2R}(x_0) \subset \Omega\backslash\omega$. Let $y_0^+ := \sup(y_0, 0)$, $y_0^- := \sup(-y_0, 0)$. Define analogously u^+ and u^-. Let Y_+ (resp. Y_-) be the (unique) solution of problem $\mathcal{P}(u^+)$ with initial datum y_0^+ (resp. $\mathcal{P}(u^-)$ with initial datum y_0^-). Then, by the comparison principle (*see* [8]), $-Y_-(x,t) \leq y(x,t) \leq Y_+(x,t)$ and $Y_+(x,t)$ (resp. $Y_-(x,t)$) ≥ 0 for any $t \in [0,T]$ and a.e. $x \in \Omega$. Then, the function Y_+ (resp. Y_-) and $\hat{y} \equiv 0$ satisfy the equation in $\mathcal{D}'(B_{2R}(x_0) \times (0,T))$ and therefore, by (2),

$$\int_{B_R(x_0)} Y_+(x,t)dx \leq C\left[\int_{B_{2R}(x_0)} (y_0^+(x) + t^\alpha R^{-\beta})dx\right]$$

for any $t \in [0,T]$. In consequence,

$$\int_{B_R(x_0)} |y(x,t)|dx \leq C\left[\int_{B_{2R}(x_0)} (|y_0(x)| + t^\alpha R^{-\beta})dx\right] \tag{5}$$

for any $t \in [0,T]$. It is clear that (5) implies an obstruction for the $L^1(\Omega)$-norm of $y(t;u)$.

The following result shows a *pointwise obstruction phenomenon.*

PROPOSITION 1. – *Let $y(\cdot\,;u)$ be a solution of $\mathcal{P}(u)$ with $\varphi(s) = |s|^{m-1}s$, $0 < m < 1$. Then, for a large class of functions $y_0 \in L^2(\Omega)$ (those satisfying* (9)*), there exist two functions $\underline{U}$, $\overline{U} \in L^\infty(0,T : L^\infty_{\text{loc}}(\Omega\backslash\omega))$ such that, for any $u \in L^2(\omega \times (0,T))$, $\underline{U}(x,t) \leq y(x,t;u) \leq \overline{U}(x,t)$ for any $t \in [0,T]$ and a.e. $x \in \Omega\backslash\omega$.*

Proof. – For any $\lambda > 0$, we take $p := 1/m$ (therefore $p > 1$) and define the function $Y_\lambda^+(x)$ (resp. $Y_\lambda^-(x)$) as the (unique) solution of

$$\begin{cases} -\Delta Y_\lambda^+ + \lambda|Y_\lambda^+|^{p-1}Y_\lambda^+ = 0 & \text{in } \Omega\backslash\omega, \\ Y_\lambda^+ = 0 & \text{on } \partial\Omega, \\ Y_\lambda^+ = \infty \quad (\text{resp. } Y_\lambda^- = -\infty) & \text{on } \partial\omega \end{cases} \tag{6}$$

(*see* e.g., [1]). Then it is easy to prove that, for any constant $C > 0$, the function

$$(7)\qquad \begin{cases} \overline{U}(x,t) := (Y_\lambda^+(x))^p[(1-m)\lambda t + C^{1-m}]^{\frac{1}{1-m}} \\ (\text{resp. } \underline{U}(x,t) := -|Y_\lambda^-(x)|^p[(1-m)\lambda t + C^{1-m}]^{\frac{1}{1-m}}) \end{cases}$$

is a solution of

$$(8)\qquad \begin{cases} \overline{U}_t - \Delta(|\overline{U}|^{m-1}\overline{U}) = 0 & \text{in } \mathcal{D}'(\Omega\backslash\omega \times (0,T)), \\ |\overline{U}|^{m-1}\overline{U} = 0 & \text{on } \Sigma, \\ \overline{U} = \infty \quad (\text{resp. } \underline{U} = -\infty) & \text{on } \partial\omega \times (0,T), \\ \overline{U}(0,x) = CY_\lambda^+(x) & \text{in } \Omega\backslash\omega. \end{cases}$$

Thus, if we assume that

$$(9)\qquad \begin{cases} \text{there exists } C > 0 \text{ and } \lambda > 0 \text{ such that} \\ -C|Y_\lambda^-(x)|^p \le y_0(x) \le C(Y_\lambda^+(x))^p \quad \text{for a.e. } x \in \Omega\backslash\omega, \end{cases}$$

by the comparison principle $\underline{U}(x,t) \le y(x,t;u) \le \overline{U}(x,t)$ for a.e. $x \in \Omega\backslash\omega$, and any $u \in L^2(\omega \times (0,T))$.

We point out that the uniqueness of solutions of (8) with general initial data may fail (in contrast with the case of nonsingular solutions of $\mathcal{P}(u)$ or singular solutions of semilinear equations). This is the case if, for instance $y_0 \equiv 0$ (for any $\lambda > 0$ the function $U_\lambda(x,t) := (m-1)(\lambda t)^{1/(1-m)}Y_\lambda^+(x)$ is a solution of (8) with zero initial value).

It is well-known (*see* [1]) that $Y_\lambda^-(x)$ and $Y_\lambda^+(x)$ behave as $C_\lambda d(x,\partial\omega)^{-2/(p-1)}$, $C_\lambda > 0$, when $x \in \Omega\backslash\omega$ is near $\partial\omega$. Then, if $p \in (1,2)$ (*i.e.* $m \in (1/2,1)$), we have that $z(x,t) = (-\Delta)^{-1}y(x,t)$ (with homogeneous Dirichlet conditions on $\partial\Omega$) satisfies $|z(x,t)| \le K(t)d(x,\partial\omega)^{-2(2-p)/(p-1)}$ for some $K(t) > 0$ and for $x \in \Omega\backslash\omega$ near $\partial\omega$. So we get:

Proposition 2. – *Let $y \in \mathcal{C}([0,T] : H^{-1}(\Omega))$ be the solution of $\mathcal{P}(u)$ with $\varphi(s) = |s|^{m-1}s$, and $m \in (1/2,1)$, $y_0 \in L^2(\Omega)$ satisfying* (9), *and $h \equiv 0$. Then we can choose $y_d \in H^{-2}(\Omega)$ such that, for any $u \in L^2(\omega \times (0,T))$, $\|y(T;u) - y_d\|_{H^{-2}(\Omega)} > \delta$ for any $\delta > 0$ small enough.*

3. An approximate controllability result when the nonlinearity is essentially linear at infinity

Theorem 2. – *Let φ be a continuous nondecreasing function such that $\varphi \in C^1(\mathbb{R}\backslash[-M_1,M_1])$, for some constant $M_1 \ge 0$, and $\varphi(0) = 0$. Assume that there exists two positive constants k and C_1 such that* (3) *and* (4) *hold. Then, if $\varphi'(s) \ge c > 0$ for a.e. $s \in \mathbb{R}$ or $h \in L^2(Q)$, problem $\mathcal{P}(u)$ satisfies the approximate controllability property on $H^{-1-\gamma}(\Omega)$ for any $\gamma > 0$.*

In order to prove Theorem 2, we take $\varepsilon > 0$ arbitrary and study the approximate controllability for the vanishing viscosity higher order problem $\mathcal{P}(u)$.

Theorem 3. – *If we assume φ like in Theorem 2, $\gamma > 0$ and $y_d \in H^{-1-\gamma}(\Omega)$, then there exists a sequence of uniformly bounded controls $v_\varepsilon \in L^2(\omega \times (0,T))$ such that the associated solutions $y(\cdot\,;v_\varepsilon)$ satisfy* (1).

Sketch of proof of Theorem 3. – Since (3) clearly implies that $\varphi'(s) \to k$ as $|s| \to \infty$, it is natural to define $\varphi_0(s) := \varphi(s) - ks$ (so that $\varphi_0'(s) \to 0$ as $|s| \to \infty$). Then, we linearize function φ_0 near a point $s_\varepsilon \in \mathbb{R}$ suitable chosen. Some elementary Calculus techniques lead to:

Lemma 2. – *Let $\varphi \in C^0(\mathbb{R})$ satisfying* (3). *Given $\varepsilon > 0$ there exists $s_\varepsilon \in \mathbb{R}$ such that*

$$(10)\qquad g_\varepsilon(s) := \frac{\varphi_0(s) - \varphi_0(s_\varepsilon)}{s - s_\varepsilon}$$

satisfies $g_\varepsilon \in L^\infty(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ *and*

$$(11) \qquad \|g_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \sqrt{\varepsilon}.$$

If, in addition, φ *satisfies* (4), *then there exists a constant* K_2, *independent of* ε *such that*

$$(12) \qquad |g_\varepsilon(s)s_\varepsilon| \leq K_2, \qquad \textit{for any} \quad \varepsilon > 0 \quad \textit{and any} \quad s \in \mathbb{R}.$$

End of the sketch of proof of Theorem 3. – Returning to our linearizing process, since $\varphi_0(s) = \varphi_0(s_\varepsilon) + g_\varepsilon(s)s - g_\varepsilon(s)s_\varepsilon$, we consider the approximate controllability for a linear problem obtained by replacing the term $\varphi(y)$ by $ky + g_\varepsilon(z)y + \varphi_0(s_\varepsilon) - g_\varepsilon(z)s_\varepsilon$, where z is an arbitrary function in $L^2(Q)$. Notice that, when $z = y$, this expression coincides with $\varphi(y)$ and that, if we denote $h_\varepsilon(z) := \Delta(\varphi_0(s_\varepsilon) - g_\varepsilon(z(t,z))s_\varepsilon)$, then $h_\varepsilon(z) \in L^\infty(0,T : H^{-2}(\Omega))$ for all $z \in L^2(Q)$ and all $\varepsilon > 0$. Now, we consider the approximate controllability property corresponding to the linear problem

$$(13) \qquad \begin{cases} y_t + \varepsilon\Delta^2 y - k\Delta y - \Delta((g_\varepsilon(z)y) = h + h_\varepsilon(z) + u_\varepsilon\chi_\omega & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$

Then, in a way similar to the proof of Theorem 1 of [5] but taking into account now Lemma 2, we deduce that, given $y_d \in H^{-1-\gamma}(\Omega)$ and $\delta > 0$, there exists a contant K, independent of ε, such that we can find solutions of (13) satisfying (1) with

$$(14) \qquad \|u_\varepsilon\|_{L^2(Q)} \leq K.$$

If we define $\Lambda_\varepsilon : L^2(Q) \to \mathcal{P}(L^2(Q))$ by $\Lambda_\varepsilon(z) := \{y_\varepsilon$ satisfying (13) and (1), with u_ε satisfying (14)$\}$, then, by Kakutani's fixed point Theorem, we deduce that, for all $\varepsilon > 0$, there exists a fixed point y_ε of Λ_ε, and y_ε is the solution of $\mathcal{P}(u)$ for some control u satisfying (14).

Proof of Theorem 2. – *First step.* Assume in addition that $\varphi \in C^1(\mathbb{R})$. For any $\varepsilon > 0$, let v_ε and y_ε be the functions given in Theorem 3. Assume $\varphi'(s) \geq c > 0$ for all $s \in \mathbb{R}$ (resp. $h \in L^2(Q)$). Since the equation of $\mathcal{P}_\varepsilon(u)$ holds on $L^2(0,T : (H^2(\Omega) \cap H_0^1(\Omega))')$, multiplying by $y_\varepsilon \in L^2(0,T : (H^2(\Omega) \cap H_0^1(\Omega))$ and applying Young (resp. Gronwall) inequality, we obtain

$$(15) \qquad \|y_\varepsilon\|_{L^\infty(0,T:L^2(\Omega))} + \int_Q \varphi'(y_\varepsilon)|\nabla y_\varepsilon|^2 dxdt \leq C.$$

Therefore, from (15) we obtain that y_ε is uniformly bounded in $L^\infty(0,T : L^2(\Omega))$ and, by the equation of $\mathcal{P}_\varepsilon(u)$, $(y_\varepsilon)_t$ is uniformly bounded in $L^\infty(0,T : H^{-4}(\Omega))$. Then, since $L^2(\Omega) \subset H^{-1}(\Omega) \subset H^{-4}(\Omega)$ with compact embeddings, we have (*see* [10]) that y_ε is relatively compact in $\mathcal{C}([0,T] : H^{-1}(\Omega))$. Further, from (15) and the boundedness of function φ' (notice that $\varphi' \in L^\infty(\mathbb{R})$ by (3)), we deduce that there exists a constant $K > 0$ independent of ε such that

$$\int_0^T \|\nabla\varphi(y_\varepsilon)\|^2_{L^2(\Omega)} dt = \int_Q \varphi'(y_\varepsilon(x,t))\varphi'(y_\varepsilon(x,t))|\nabla(y_\varepsilon(x,t))|^2 dxdt < K.$$

Thus, there exist $y \in L^\infty(0,T : L^2(\Omega))$ and $\zeta \in L^2(0,T : H_0^1(\Omega))$ (recall that $\varphi(0) = 0$) such that $y_\varepsilon \to y$ strongly in $\mathcal{C}([0,T] : H^{-1}(\Omega))$ and $\varphi(y_\varepsilon) \rightharpoonup \zeta$ weakly in $L^2(0,T : H_0^1(\Omega))$. But the operator $Au := -\Delta\varphi(u)$, $D(A) := \{u \in H^{-1}(\Omega) \cap L^1(\Omega) : \varphi(u) \in H_0^1(\Omega)\}$, is a maximal monotone operator on the space $H^{-1}(\Omega)$ (*see* [2]). Thus, the extension operator $\mathcal{A}$ of A on $L^2(0,T:H^{-1}(\Omega))$ is also a maximal monotone operator (*see* [3], Example 2.33). Finally, as any maximal monotone operator is strongly-weakly closed (*see* [3], Proposition 2.5), we obtain that $\zeta = \varphi(y)$ in $L^2(0,T:H_0^1(\Omega))$. Moreover, from the uniform boundedness we have that $v_\varepsilon \rightharpoonup v$ weakly in $L^2(\omega \times (0,T))$, with

$$\|v\|_{L^2(\omega\times(0,T))} \leq K. \tag{16}$$

Then we deduce that $y \in \mathcal{C}([0,T]:H^{-1}(\Omega))$ is a solution of $\mathcal{P}(v)$ and that $\|y(\cdot,T)-y_d\|_{H^{-1-\gamma}(\Omega)} \leq \delta$.

Second step. – We approximate φ by $\varphi_n \in C^1(\mathbb{R})$, φ_n nondecreasing, satisfying (3) and (4) with the same constants k, C_1, C_2, and, M_1 that the ones for φ. Then the controls v_n, built as in the first step, are uniformly bounded (see (16)) and so the conclusion comes from well-known results showing that $y_n \rightharpoonup y$ in $\mathcal{C}([0,T]:H^{-1}(\Omega))$ when $\varphi_n \rightharpoonup \varphi$ (*see* e.g. [4]).

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