Journal of Mathematical Analysis and Applications **233**, 713–729 (1999)
Article ID jmaa.1999.6335, available online at http://www.idealibrary.com on IDEAL®

Time Periodic Solutions for a Diffusive Energy Balance Model in Climatology

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Submitted by Chia Ven Pao

Received November 3, 1997

We prove the existence of a periodic solution to the problem

$$u_t - \Delta_p u + R_e(x,u) \in \mu Q(x,t)\beta(u) \quad \text{in} \quad \mathscr{M} \times \mathbb{R},$$

assumed $p \geq 2$, $\mathscr{M}$ a compact connected and oriented bidimensional Riemannian manifold without boundary, $\beta(u)$ a bounded maximal monotone graph (the coalbedo), $Q(x,t)$ a time periodic function (the incoming solar radiation flux) and R_e a time independent strictly increasing function of the surface temperature u (the Earth emitted energy).

1. INTRODUCTION

We consider a time evolution model for the Earth surface temperature obtained via an energy balance. This type of climate model, independently introduced in 1969 by Budyko [6] and Sellers [17], has a spatial global nature and involves a relatively long-time scale. Our study concerns the

*Partially supported by G.N.A.F.A.-C.N.R. M.U.R.S.T., 40%.
†Partially supported by the DGICYT (Spain), Project No. PB96-0583.

0022-247X/99 $30.00

existence of periodic solutions of the nonlinear parabolic problem

$$(P)\quad u_t - \Delta_p u + R_e(x,u) \in \mu Q(x,t)\beta(u) \quad \text{in} \quad \mathcal{M}\times\mathbb{R},$$

where $p \geq 2$ and $\mathcal{M}$ is a compact connected and oriented bidimensional Riemannian manifold without boundary (e.g., $\mathcal{M} = S^2$ the unit sphere in $\mathbb{R}^3$). Notice the missing boundary conditions in (P) once $\mathcal{M}$ has no boundary. We use the notation $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where the differential operators div and ∇ must be understood in the sense of the Riemannian metric on $\mathcal{M}$ (see, e.g., [2, 12, and 20]). We point out that although in Budyko [6] and Sellers [17] the diffusion operator was linear (i.e., $p = 2$), the quasilinear case $p = 3$ was proposed in [19].

Function $Q(x,t)$ describes the *incoming solar radiation flux*. The general assumption $Q(x,t) \geq 0$ allows to consider the polar night phenomena (time where $Q(x,t) = 0$ for some subsets of the manifold $\mathcal{M}$). The term $\mu Q(x,t)\beta(u)$ corresponds to the fraction of solar energy absorbed by the Earth, μ is a positive constant representing the so called *solar constant*, and the *coalbedo* $\beta(u)$ represents the fraction of the absorbed energy: a nondecreasing function of u of the type $\beta(u) = 0.7$ if $u > -10 + \epsilon$, $\beta(u) = 0.4$ if $u < -10 - \epsilon$ with $\epsilon \geq 0$ assumed the temperature u given in Celsius degrees. The term β was assumed to be multivalued (i.e., with discontinuous sections) at $u = -10$ by Budyko [6] and a Lipschitz function by Sellers [17]. The term R_e represents the *emitted energy by the Earth* to the outer space and it is usually assumed to be an increasing function on u. Function R_e can be expressed according the Newton cooling law as $Bu + C$, with B and C positive constants (Budyko [6]) or the Stefan-Boltzman law $R_e(t,x,u) = \sigma u^4$ (this corresponds to when u is measured in degrees Kelvin and so $u > 0$) (Sellers [17]).

The consideration of the periodicity of the forcing term is motivated by the seasonal variation of the incoming solar radiation flux during one natural year. Without loss of generality we can assume that the period τ is given by $\tau = 1$. Our formulation corresponds to an ideal situation in which the Earth emitted energy R_e is stationary (i.e., time independent). Our results can be easily adapted to the case in which R_e depends 1-periodically on the time.

As we shall explain later, the treatment of problem (P) must be different according the multivalued or singlevalued character of the coalbedo term $\beta(u)$ (it turns out that the different structure assumed on the term R_e in Budyko [6] and Sellers [17] is not relevant for our purposes so that in both cases $R_e(x,u)$ is a strictly increasing function of u).

Throughout this paper we shall assume the following structural conditions:

(H_Q) $Q \in C^1(\mathbb{R}; C(\mathcal{M}))$, $Q \geq 0$, $Q(x,\cdot)$ is a 1-periodic function for any fixed $x \in \mathcal{M}$.

(H_β) β is a bounded maximal monotone graph of $\mathbb{R}^2$ such that $0 < m \leq z \leq M$, for any $z \in \beta(s)$ and for any $s \in \mathbb{R}$.

(H_R) $R_e \in C(\mathcal{M}\times\mathbb{R})$, $R_e(x,\cdot)$ is strictly an increasing function for $x \in \mathcal{M}$, $R_e(x,0) \geq 0$, $|R_e(x,s)| \geq B|s|^r$, for any $x \in \mathcal{M}$ and for any s with $|s|$ large enough, for some $r \geq 1$, $B > 0$.

In what follows we shall denote by the *Sellers type model* the case in which β is assumed to be a Lipschitz approximation of a multivalued Heaviside type graph; more precisely when β satisfies (H_β) as well as

$$\beta(\cdot) \text{ is a Lipschitz functions.} \tag{1}$$

By *Budyko type model* we shall mean the general case in which β merely satisfies (H_β).

We start our study of the existence of periodic solutions by considering the Sellers type model. Our existence result will be obtained by studying the *Poincaré map* F associated to the Cauchy problem

$$(CP)\begin{cases} u_t - \Delta p_u + R_e(x,u) = \mu Q(x,t)\beta(u) & \text{in } \mathcal{M}\times(0,T), \\ u(x,0) = u_0(x) & x \in \mathcal{M}, \end{cases}$$

where $T \geq 1$ is arbitrarily fixed and

$$u_0 \in L^\infty(\mathcal{M}) \tag{2}$$

is suitably chosen. Since (CP) degenerates, where $\nabla u = 0$ (when $p > 2$), it is well-known that classical solutions may not exist (see, e.g., [11] for the case $\mathcal{M} = \Omega$ an open set in $\mathbb{R}^N$). Because of that, we shall deal with weak solutions of (CP). The existence of (bounded) weak solutions can be proved, as in [9] and [12], by means of a fixed point argument. The natural *energy space* associated with (CP) is the one defined by $V := \{w \in L^2(\mathcal{M}), \nabla w \in L^p(T\mathcal{M})\}$, where $T\mathcal{M}$ denotes the tangent space to $\mathcal{M}$ (see, e.g., [2, 12, and 20]). The assumption (1) is enough to prove the uniqueness of a bounded weak solution for (CP).

In order to prove the existence of periodic solutions of (P) for the Sellers type model, we first construct a constant subsolution $\underline{u}$ and a constant supersolution $\overline{u}$ of (CP). Then, we consider the Poincaré map F associated to (CP), i.e. the operator assigning, to every initial data of the ordered interval $[\underline{v}, \overline{u}] := \{w \in L^\infty(\mathcal{M}), \underline{v} \le w(x) \le \overline{u}$ a.e. $x \in \mathcal{M}\}$, the solution of (CP) evaluated after 1-period. We shall prove that F is a continuous, compact, and pointwise increasing map. Then, using the Schauder fixed point theorem, we obtain the existence of, at least, one fixed point for F. Obviously, this fixed point is a periodic solution of problem (P). In addition, we shall prove that (P) has a smallest and greatest periodic solution among the ones taking values in the interval $[\underline{v}, \overline{u}]$.

The existence of a periodic solution for the Budyko type model needs some different arguments. The main difficulty comes from the fact that, in general, the uniqueness of solution for the associated Cauchy problem fails (see [9] and [12]). So, the Poincaré map cannot be well defined as a univalued operator. Due to that we shall follow a different strategy. The existence of a periodic solution will be proved, this time, by applying a variant of the Schauder-Tychonoff fixed point theorem for a suitable multivalued operator $\mathcal{S}$ defined on a convex and weakly compact set of $L^p_{per}(\mathbb{R}; L^2(\mathcal{M}))$ (the space of measurable functions from $\mathbb{R}$ into $L^2(\mathcal{M})$ which are 1-periodic and whose restrictions to $(0,1)$ belong to $L^p((0,1); L^2(\mathcal{M}))$.

Finally, in the last part, we investigate the structure of the set of periodic solutions of (P) with respect to the parameter μ. We denote by Σ the set of pairs $(\mu, u) \in \mathbb{R}^+ \times C_{per}(\mathbb{R}; L^2(\mathcal{M}))$, where u is a 1-periodic solution of (P), that is,

$$\Sigma = \{(\mu, u) : \mu \ge 0 \text{ and } u \text{ is a 1-periodic solution of } (P)\}.$$

Our goal is to describe qualitatively the solution set Σ in the space $\mathbb{R}^+ \times C_{per}(\mathbb{R}; L^2(\mathcal{M}))$. We show that this set has a closed, connected and unbounded component containing $(0, u^0)$, where $u^0 \in V$ is the unique solution of the stationary equation

$$-\Delta_p u + R_e(x, u) = 0 \text{ in } \mathcal{M}.$$

Moreover, if $Q > 0$, under a physically reasonable additional assumption on β we prove that there exist $\mu_1 > 0$ and a continuous and increasing curve $\sigma: [\mu_1, \infty) \to C_{per}(\mathbb{R}; L^2(\mathcal{M}))$ such that $\Sigma \cap [\mu_1, \infty) \times C_{per}(\mathbb{R}; L^2(\mathcal{M}))$ is the graph of the curve σ. Our results extend previous works related with Sellers model associated to the linear diffusion case $(p = 2)$ (see [13] and [14]).

2. ON THE SELLERS TYPE MODELS

2.1. *The Cauchy Problem. Construction of Sub and Supersolutions*

We start by considering the Cauchy problem (CP). The notion of weak solution we shall use is the following

DEFINITION 1. For a *bounded weak solution* to (CP) we mean a function $u \in C([0,T]; L^2(\mathcal{M})) \cap L^p(0,T; V) \cap L^\infty((0,T) \times \mathcal{M})$, such that

$$\int_{\mathcal{M}} u(x,T) v(x,T)\, dA - \int_0^T \langle v_t(x,t), u(x,t) \rangle_{V' \times V}\, dt + \int_0^T \int_{\mathcal{M}} |\nabla u|^{p-2} \nabla u \cdot \nabla u \, dA dt + \int_0^T \int_{\mathcal{M}} R_e(x,u) v \, dA dt = \int_0^T \int_{\mathcal{M}} \mu Q(x,t) \beta(u(x,t)) v(x,t)\, dA dt + \int_{\mathcal{M}} u_0(x) v(x,0)\, dA,$$

$\forall v \in L^p(0,T; V) \cap L^\infty((0,T) \times \mathcal{M})$ such that $v_t \in L^{p'}(0,T; V')$, where $\langle \cdot, \cdot \rangle_{V' \times V}$ denotes the pairing of duality on $V' \times V$ and $\cdot$ the scalar product in the tangent space $T\mathcal{M}$.

Notice that from the above definition we deduce that any bounded weak solution verifies that $u_t \in L^{p'}(0,T; V')$.

By using that the operator $u \to -\Delta_p u + R_e(x,u)$ is m-accretive in $L^2(\mathcal{M})$ (see, e.g., [4]) and the assumption (H_β), it is easy to prove the existence of, at least, a bounded weak solution of (CP). This was done in [12] by means of the application of an abstract compactness result in the case of $Q(x,t) = Q(x)$. Obvious modifications allows to extend their result to the case under consideration.

The uniqueness of the bounded weak solution to (CP) for the Sellers model is obtained using the assumption (1). Although this is a more or less well-known result we present here a detailed proof.

PROPOSITION 1. *Assume* (1), (2) *and* $u_{0,i} \in L^\infty(\mathcal{M})$, $i = 1,2$. *Let* u_i *be any bounded weak solution of* (CP) *corresponding to* $u_0 = u_{0,i}$, *then for any* $t \in [0,T]$ *we have*

$$\int_{\mathcal{M}} |(u_1(x,t) - u_2(x,t))^+|^2\, dA \le \exp(LT) \int_{\mathcal{M}} |(u_{0,1}(x) - u_{0,2}(x))^+|^2\, dA. \tag{3}$$

In particular, if $u_{0,1} \le u_{0,2}$ *on* $\mathcal{M}$ *then* $u_1 \le u_2$ *on* $\mathcal{M} \times [0,T]$. *So, there is a unique bounded weak solution of* (CP).

Proof. Since $v = (u_1 - u_2)^+$ satisfies that $v \in L^p(0,T;V) \cap L^\infty((0,T) \times \mathscr{M})$ and $v_t \in L^{p'}(0,T;V')$ we can take v as a test function. By using (H_R) and (1) the function $s \to R_e(x,u) - \mu Q(x,t)\beta(u) + Ls$ is increasing for some $L > 0$. On the other hand, the operator $u \to -\Delta_p u$ is T-monotone in $L^2(\mathscr{M})$ (see, e.g., [4] for $\mathscr{M} = \Omega$ an open set in $\mathbb{R}^N$ and [12] for a general compact connected and oriented bidimensional Riemannian manifold without boundary $\mathscr{M}$). Then, since

$$\left\langle (u_1 - u_2)_t, (u_1 - u_2)^+ \right\rangle_{V'V} = \frac{1}{2}\frac{d}{dt}\int_{\mathscr{M}} |(u_1(x,t) - u_2(x,t))^+|^2 \, dA$$

(see, e.g., [3]), we obtain that

$$\begin{aligned} &\int_{\mathscr{M}} |(u_1(x,t) - u_2(x,t))^+|^2 \, dA \\ &\quad \le \int_{\mathscr{M}} |(u_{0,1}(x) - u_{0,2}(x))^+|^2 \, dA \\ &\qquad + L\int_0^t \int_{\mathscr{M}} |(u_1(x,s) - u_2(x,s))^+|^2 \, dA\,ds. \end{aligned}$$

Estimate (3) follows now from Gronwall lemma. The rest of the conclusions trivially come from (3). ∎

Let us construct, now, some constant sub and supersolutions of (CP) (the definition of sub and supersolutions to (CP) follows usual modifications of the above definition of bounded weak solution). Thanks to (H_Q) we can assume that

$$Q_1(x) \le Q(x,t) \le Q_2(x), \quad \text{with} \quad Q_1, Q_2 \in C(\mathscr{M}), Q_1 \ge 0 \text{ and } Q_2 \ge 0. \tag{4}$$

Consider now the stationary problems

$$(SP)_1 \qquad -\Delta_p v + R_e(x,v) = \mu m Q_1(x) \qquad \text{in } \mathscr{M},$$

$$(SP)_2 \qquad -\Delta_p u + R_e(x,u) = \mu M Q_2(x) \qquad \text{in } \mathscr{M},$$

where m and M are given in (H_β). We look for constant subsolutions and supersolutions of those problems. From assumption (H_R) it is clear that there exists $\bar{u} > 0$ such that

$$R_e(x,\bar{u}) \ge \mu M \bar{Q}_2 \ge \mu M Q_2(x), \quad \text{for any } x \in \mathscr{M},$$

where $\tilde{Q}_2 := \max\{Q_2(x), x \in \mathscr{M}\}$. Analogously, there exists $\underline{v} \le 0$ such that

$$R_e(x,\underline{v}) \le \mu m \tilde{Q}_1 \le \mu m Q(x), \quad \text{for any } x \in \mathscr{M}$$

with $\tilde{Q}_1 := \min\{Q_1(x), x \in \mathscr{M}\}$.

Let us introduce the notation

$$[\underline{h}, \bar{h}] = \{w \in L^2(\mathscr{M}) : \underline{h}(x) \le w(x) \le \bar{h}(x), \text{a.e. } x \in \mathscr{M}\}$$

for any given functions $\underline{h}, \bar{h} \in L^2(\mathscr{M})$. As consequence of Proposition 1 we have

PROPOSITION 2. *Assume* (1) *and let* $u_0 \in [\underline{v}, \bar{v}]$. *Then the bounded weak solution u of (CP) satisfies that* $u(\cdot,t) \in [\underline{v}, \bar{u}]$, *for any* $t \in [0,T]$.

Remark 1. Notice that if the temperature is in degrees Kelvin we can take $\underline{v} = 0$. If the temperature is measured in Celsius degrees, $R_e(x,u) = Bu + A$ and we assume $A > 10B$ then we can take

$$\underline{v} = -\frac{A}{B} < -10.$$

2.2. *Existence of periodic solutions for the Sellers type model via the Poincaré map*

Concerning problem (P) we introduce the following definition which is also valid for the Budyko type model

DEFINITION 2. For a 1-periodic bounded weak solution to (P), we mean a function $u \in C(\mathbb{R}; L^2(\mathscr{M})) \cap L^\infty(\mathbb{R} \times \mathscr{M})$ such that $u \in L^p_{loc}(\mathbb{R}; V)$, $u_t \in L^{p'}_{loc}(\mathbb{R}; V')$, $u(x,t+1) = u(x,t)$, a.e. $x \in \mathscr{M}$ and for any $t \in \mathbb{R}$ and there exists $z \in L^\infty(\mathbb{R} \times \mathscr{M})$, with $z(x,t) \in \beta(u(x,t))$ for a.e. (x,t) in $\mathbb{R} \times \mathscr{M}$, such that $\forall\, I := [t_0, t_1]$

$$\begin{aligned} &\int_I \langle u_t, v \rangle_{V' \times V} \, dt + \int_I\int_{\mathscr{M}} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dA\,dt + \int_I\int_{\mathscr{M}} R_e(x,u) v \, dA\,dt \\ &\quad = \int_I\int_{\mathscr{M}} \mu Q(x,t) z(x,t) v(x,t) \, dA\,dt \end{aligned}$$

$\forall v \in L^p(I;V) \cap L^\infty(I \times \mathscr{M})$. (Notice that $z(x,t) = \beta(u(x,t))$ for the Sellers type model).

We have

THEOREM 3. *Assume* (1). *Then there exists at least a* 1*-periodic bounded solution of* (P).

Proof. Recalling that in that case (CP) has a unique bounded weak solution, we can define the Poincaré map F by $F(u_0(\cdot)) = u(\cdot, 1)$. In order to apply the Schauder fixed point theorem on the space $L^2(\mathscr{M})$, we need to select a closed and convex set $K \subset L^2(\mathscr{M})$ such that

(i) $F(K) \subset K$;

(ii) $F_{|K}$ is continuous;

(iii) $F(K)$ is relatively compact in $L^2(\mathscr{M})$.

We take $K := [\underline{u}, \overline{u}]$. Obviously K is a closed, convex, and non empty set of $L^2(\mathscr{M})$. Now, property (i) follows from Proposition 2 since we have shown that $F([\underline{u}, \overline{u}]) \subset [\underline{u}, \overline{u}]$. Let us check property (ii). We have

LEMMA 4. *Assume* (1) *and let* $u_{0,n}$, $u_0 \in K$ *be such that* $u_{0,n} \to u_0$ *in* $L^2(\mathscr{M})$ *as* $n \to \infty$. *Then, if* u_n, u *are the corresponding solutions of* (*CP*) *we have that* $u_n(\cdot, t) \to u(\cdot, t)$ *in* $L^2(\mathscr{M})$ *for any* $t \in [0, T]$.

Proof. It is again, a consequence of the fact that the operator $u \to -\Delta_p u$ is accretive in $L^2(\mathscr{M})$. The idea of the proof is stated in the following. Multiplying by a regularization $p_\varepsilon(u_n - u)$ of $(u_n - u)_+$ such that $p_\varepsilon(u_n - u) \in L^p(0, T : V)$ and integrating on $\mathscr{M} \times (0, t)$ we arrive at

$$\begin{aligned}
&\int_0^t \langle (u_n - u)_t, p_\varepsilon(u_n - u) \rangle_{V' \times V}\, ds \\
&\quad - \int_0^t \int_{\mathscr{M}} \left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) \cdot \nabla p_\varepsilon(u_n - u)\, dA ds \\
&\quad = \int_0^t \int_{\mathscr{M}} [\, \mu Q(x,t) \beta(u_n) - R_e(x, u_n) \\
&\qquad\qquad - \mu Q(x,t) \beta(u) + R_e(x, u))]\, p_\varepsilon(u_n - u)\, dA ds.
\end{aligned}$$

Passing to the limit, as $\epsilon \to 0$, we get

$$\begin{aligned}
\int_{\mathscr{M}} |(u_n(t) - u(t))^+|^2\, dA &\le \int_{\mathscr{M}} |(u_{0,n} - u_0)^+|^2\, dA \\
&\quad + L \int_0^t \int_{\mathscr{M}} |(u_n - u)^+|^2\, dA ds.
\end{aligned}$$

The Gronwall lemma gives us

$$\int_{\mathscr{M}} |(u_n(t) - u(t))^+|^2\, dA \le \exp(LT) \int_{\mathscr{M}} |(u_{0,n} - u_0)^+|^2\, dA$$

and changing the roles of u with u_n we obtain

$$\int_{\mathscr{M}} |u_n(t) - u(t)|^2\, dA \le \exp(LT) \int_{\mathscr{M}} |u_{0,n} - u_0|^2\, dA.$$

So the conclusion holds. ∎

Proof of Theorem 3 (*Continuation*). In order to show that $F(K)$ is relatively compact in $L^2(\mathscr{M})$ we shall start by proving some a priori estimates. By taking u as test function in Definition 1 (written now for $T = 1/2$) we get that

$$\frac{1}{2} \int_{\mathscr{M}} u(x, 1/2)^2\, dA + \int_0^{1/2} \int_{\mathscr{M}} |\nabla u|^p\, dA\, ds \le C_1 \tag{5}$$

for any bounded weak solution with $u_0 \in K$, for some $C_1 > 0$. On the other hand,

$$\left.\begin{array}{l}\text{the function } t \to \int_{\mathscr{M}} |\nabla u|^p\, dA \text{ is continuous from } [\delta, 1/2] \text{ into } \mathbb{R}, \\ \text{for any } \delta \in (0, 1/2).\end{array}\right\} \quad (\mathrm{R})$$

Although stronger regularity has been shown by E. Di Benedetto and other authors (see, e.g., Di Benedetto [7]) a direct proof of (R) can be obtained from the abstract theory of evolution equations for maximal monotone operators in Hilbert spaces. Indeed, from the uniqueness of solutions for the (CP), function u coincides with the solution of the equation

$$\begin{cases} u_t(t) + Au(t) = h(t), & t \in (0, 1/2), \quad \text{in } L^2(\mathscr{M}), \\ u(0) = u_0, \end{cases}$$

where the operator A: $D(A) \subset L^2(\mathscr{M}) \to L^2(\mathscr{M})$ is given by $Au := -\Delta_p u$ and $D(A) = \{u \in V \colon Au \in L^2(\mathscr{M})\}$ and where $h(t) := -R_e(\cdot, u(t)) + \mu Q(\cdot, t) \beta(u(t))$. Since A is the maximal monotone operator in $L^2(\mathscr{M})$ given by $A = \partial \varphi$ with

$$\varphi(u) = \begin{cases} \dfrac{1}{p} \displaystyle\int_{\mathscr{M}} |\nabla u|^p\, dA & \text{if } u \in V, \\ +\infty & \text{otherwise,} \end{cases}$$

(see, e.g., Proposition 1 of Díaz and Tello [12]), $u_0 \in L^\infty(\mathscr{M}) \subset L^2(\mathscr{M}) = \overline{D(A)}$ and $h \in L^\infty(0, 1/2 : L^\infty(\mathscr{M})) \subset L^2(0, 1/2 : L^\infty(\mathscr{M}))$ we can apply the abstract results available in the literature. In particular, Theorem 3.6 of Brezis [5] shows that the function $t \to \varphi(u(t))$ is continuous from $[\delta, 1/2]$ into $\mathbb{R}$, for any $\delta \in (0, 1/2)$. This is, exactly, the regularity (R) mentioned before.

Then, from (5), (R) and the mean value theorem we deduce that necessarily there is $\tau \in (0, 1/2)$ such that

$$\int_{\mathscr{M}} |\nabla u(x,\tau)|^p \, dA \le C_1. \tag{6}$$

Now, we multiply the equation by u_t and integrating from τ to 2 (this process can be justified via Definition 1 by means of standard regularizing methods). Then we get that

$$\begin{aligned}
&\int_\tau^2 \int_{\mathscr{M}} u_t(x,t)^2 \, dA\,dt + \frac{1}{p}\int_\tau^2 \frac{d}{dt}\left(\int_{\mathscr{M}} |\nabla u|^p \, dA\right) dt \\
&\quad + \int_\tau^2 \frac{d}{dt}\left(\int_{\mathscr{M}} j(x,u(x,t)) \, dA\right) dt \\
&\quad = \mu \int_\tau^2 \int_{\mathscr{M}} Q(x,t)\beta(u(x,t))u_t(x,t) \, dA\,dt \\
&\quad \le \frac{1}{2}\int_\tau^2 \int_{\mathscr{M}} u_t(x,t)^2 \, dA\,dt + C_2.
\end{aligned}$$

for some $C_2 > 0$, where we have applied Young's inequality and assumptions (H_Q) and (H_β). Here j is a nonnegative primitive of R_e, i.e.,

$$j(x,r) = \int_{r_0}^r R_e(x,s)\, Ds, \qquad |r| \le \|u\|_{L^\infty(\mathscr{M}\times(0,\infty))},$$

for some r_0, so that $j(x,r) \ge 0$ for any $|r| \le \|u\|_{L^\infty(\mathscr{M}\times(0,\infty))}$. Using (6) we conclude that

$$\operatorname*{ess\,sup}_{t\in(\tau,2)} \int_{\mathscr{M}} |\nabla u(x,t)|^p \, dA \le C_3$$

for some $C_3 > 0$. Then, if we define

$$\mathscr{F} = \{u \in L^p(\tau, 2 : V) \text{ such that } u \text{ is the bounded weak solution of } (CP) \text{ on } \mathscr{M}\times(0,2) \text{ with } u_0 \in K\},$$

then we have shown that $\mathscr{F}$ is bounded in $L^\infty(\tau, 2 \colon V)$. Moreover, from the equation we have that the set

$$\mathscr{F}_t = \{u_t \colon u \in \mathscr{F}\}$$

is bounded in $L^{p'}(\tau, 2 : V')$. Then, by Corollary 4 of Simon [18] we have that $\mathscr{F}$ is relatively compact in $C([\tau, 2] \colon L^2(\mathscr{M}))$ which implies that $F(K)$

is relatively compact in $L^2(\mathscr{M})$. Then, by the Schauder fixed point theorem there exists a fixed point for the Poincaré map F. This fixed point is, obviously, a periodic solution for (P).

Remark 2. In the special case in which $\beta(s)$ is a constant, the uniqueness of the periodic solution can be obtained by using that the operator $-\Delta_p u$ is a monotone operator in $L^2(\mathscr{M})$ and that $R_e(\cdot, u)$ is strictly increasing in u. Nevertheless, if $\beta(s)$ is not constant, the uniqueness of the periodic solution for the Sellers type model may fail since every stationary solution of the (CP) is a periodic solution and by the results of Díaz, Hernández, and Tello [10] or Arcoya, Díaz, and Tello [1] we know that for suitable β there is more than one stationary solution. ∎

Remark 3. The above remark on the nonuniqueness of periodic solutions make relevant the following fact: we can show the existence of the smallest (respectively greatest) periodic solution of (P) such that $u(\cdot, t) \in [\underline{v}, \bar{u}]$. Indeed we consider the Cauchy problems

$$(\underline{CP})\begin{cases} z_t - \Delta_p z + R_e(x,z) = \mu Q(x,t)\beta(z) & \text{in } \mathscr{M}\times(0,T), \\ z(x,0) = \underline{v} & \text{in } \mathscr{M}, \end{cases}$$

and

$$(\overline{CP})\begin{cases} w_t - \Delta_p w + R_e(x,w) = \mu Q(x,t)\beta(w) & \text{in } \mathscr{M}\times(0,T), \\ w(x,0) = \bar{u} & \text{in } \mathscr{M}, \end{cases}$$

from Proposition 1 we see that $\underline{v} \le F(\underline{v})$ and $F(\bar{u}) \le \bar{u}$. Defining, by recurrence, the sequences

$$z_1 = F(\underline{v}), \ldots, z_n = F(z_{n-1}), \ldots$$

and

$$w_1 = F(\bar{u}), \ldots, w_n = F(w_{n-1}), \ldots$$

and arguing, for instance as in Pao [15], it is easy to see that $\{z_n\}$ and $\{w_n\}$ are monotone sequences such that

$$\underline{v} \le z_1 \le \cdots \le z_n \le w_n \le \cdots \le w_1 \le \bar{u},$$

and

$$\|z_n(1)\|_{L^\infty(\mathscr{M})} \le C, \qquad \|w_n(1)\|_{L^\infty(\mathscr{M})} \le C.$$

So, there exist their pointwise limits

$$\lim_{n\to\infty} z_n(x,1) = \underline{z}(x,1), \tag{7}$$

$$\lim_{n\to\infty} w_n(x,1) = \overline{w}(x,1). \tag{8}$$

The convergence in (7) and (8) takes place in $L^2(\mathscr{M})$. Since F is a continuous map, $\underline{z}(x,1) = \lim z_n(x,1) = \lim F(z_{n-1}) = F(\underline{z}(x,1))$ and $\overline{w}(x,1) = F(\overline{w}(x,1))$. Thus, $\underline{z}(x,t)$ and $\overline{w}(x,t)$ are the smallest, respectively the greatest, periodic solutions of (P) in the ordered interval $[\underline{v}, \overline{u}]$.

3. EXISTENCE OF PERIODIC SOLUTIONS FOR BUDYKO TYPE MODELS

Now we shall prove the existence of a periodic solution for the general case in which β merely satisfies (H_β).

THEOREM 5. *Under the assumptions (H_Q), (H_β), and (H_R) there exists, at least, a 1-periodic bounded weak solution of (P).*

Proof. Consider the Cauchy problem associated to the operator A: $D(A) \subset L^2(\mathscr{M}) \to L^2(\mathscr{M})$, given by $Au := -\Delta_p u + R_e(\cdot, u)$, $D(A) = \{u \in V\colon Au \in L^2(\mathscr{M})\}$,

$$(P_h)\begin{cases} u_t(t) + Au(t) = h(t), & t \in \mathbb{R}, \text{in } L^2(\mathscr{M}), \\ u(0) = u(1). \end{cases}$$

Since A is a maximal monotone operator in $L^2(\mathscr{M})$ (see [12]) we can apply the abstract results of [5] and so, for any 1-periodic function $h \in L^2_{per}(\mathbb{R}; L^2(\mathscr{M}))$, (P_h) has at least a periodic solution in the space $C_{per}(\mathbb{R}; L^2(\mathscr{M}))$. Moreover, as mentioned in Remark 2, this solution is the unique periodic solution since R_e is strictly increasing. Define now the set

$$G := \{z \in L^p_{per}(\mathbb{R}; L^\infty(\mathscr{M}))\colon \|z(t)\|_{L^\infty(\mathscr{M})} \le C_0, \text{a.e. } t \in \mathbb{R}\}$$

for some constant C_0. The set G is convex and weakly compact in $L^p_{per}(\mathbb{R}; L^2(\mathscr{M}))$. Using that for any $\lambda > 0$ the operator $(I + \lambda A)^{-1}$: $L^2(\mathscr{M}) \to L^2(\mathscr{M})$ is relatively compact (see [12]) then by Proposition 2.2.2 and Corollary 2.3.2 of [21], the operator J defined by

$$J\colon G \to C_{per}(\mathbb{R}; L^2(\mathscr{M}))$$
$$z \to v,$$

where v is the periodic solution of (P_h) with $h = z$, is sequentially continuous from $L^p_{per}(\mathbb{R}; L^2(\mathscr{M}))$ endowed with its weak topology into $C_{per}(\mathbb{R}; L^2(\mathscr{M}))$ endowed with its strong topology. We also introduce the operator

$$\mathscr{F}\colon L^p_{per}(\mathbb{R}; L^2(\mathscr{M})) \to 2^{L^p_{per}(\mathbb{R}; L^2(\mathscr{M}))}$$
$$\{h \in L^p_{per}(\mathbb{R}; L^2(\mathscr{M}))\colon v \to h(x,t) \in \mu Q(x,t)\beta(v(x,t)) \text{ for a.e. } x \in \mathscr{M} \text{ and for a.e. } t \in \mathbb{R}\}.$$

Finally, we define

$$\mathscr{L}\colon G \to 2^{L^p_{per}(\mathbb{R}; L^2(\mathscr{M}))},$$

$$\mathscr{L}(z) = \{h \in L^p_{per}(\mathbb{R}; L^2(\mathscr{M}))\colon h(t) \in \mathscr{F}(J(z)(t)) \text{ in } L^2(\mathscr{M}) \text{ for a.e. } t \in \mathbb{R}\}.$$

We want to apply a variant of the Schauder-Tychonoff fixed point theorem (see [21]), to $\mathscr{L}$. We must prove that graph $(\mathscr{L})$ is weakly $\times$ weakly sequentially closed in $L^p_{per}(\mathbb{R}; L^2(\mathscr{M})) \times L^p_{per}(\mathbb{R}; L^2(\mathscr{M}))$. Let $(z,h) \in \overline{graph(\mathscr{L})}^{weak \times weak}$ and $(z_n, h_n) \in graph(\mathscr{L})$, such that

$$z_n \rightharpoonup z \quad \text{in } L^p_{per}(\mathbb{R}; L^2(\mathscr{M}))$$
$$h_n \rightharpoonup h \quad \text{in } L^p_{per}(\mathbb{R}; L^2(\mathscr{M})).$$

From the above property of J we conclude that

$$J(z_n) \to J(z) \quad \text{in } C_{per}(\mathbb{R}; L^2(\mathscr{M})).$$

Since the graph$(\mathscr{F})$ is strongly $\times$ weakly sequentially closed in $L^2(\mathscr{M}) \times L^2(\mathscr{M})$, and since $(J(z_n)(t), h_n(t)) \in graph(\mathscr{F})$ for a.e. $t \in \mathbb{R}$ we get

$$(J(z)(t), h(t)) \in graph(\mathscr{F}) \quad \text{for a.e. } t \in \mathbb{R}$$

i.e., $h(t) \in \mathscr{F}(J(z)(t))$ for a.e. $t \in \mathbb{R}$ which implies $h \in \mathscr{L}(z)$. Thus, $\mathscr{L}$ has at least a fixed point in G, $u \in \mathscr{L}(u)$ which is a 1-periodic weak solution of (P). ∎

4. ON THE STRUCTURE OF THE PERIODIC SOLUTIONS SET

Let us study the structure of the set of periodic solutions of (P) with respect to the parameter μ. As mentioned at the introduction, we denote by Σ the set of pairs $(\mu, u) \in \mathbb{R}^+ \times C_{per}(\mathbb{R}; L^2(\mathcal{M}))$, where u is a 1-periodic solution of (P).

THEOREM 6. *The set Σ has a closed, connected and unbounded component containing $(0, u^0)$, where $u^0 \in V$ is the unique solution of the stationary equation*

$$-\Delta_p u + R_e(x,u) = 0 \text{ in } \mathcal{M}.$$

Moreover, if $Q > 0$ and β is a bounded maximal monotone graph of $\mathbb{R}^2$ such that there exist two real numbers M and ϵ such that $\beta(r) = \{M\}$ for any $r \in (-10 + \epsilon, +\infty)$ then there exist $\mu_1 > 0$ and a continuous and increasing curve σ: $[\mu_1, \infty) \to C_{per}(\mathbb{R}; L^2(\mathcal{M}))$ such that $\Sigma \cap [\mu_1, \infty) \times C_{per}(\mathbb{R}; L^2(\mathcal{M}))$ is the graph of the curve σ.

Proof. We start by considering the case of the additional assumption (1). The first part is a consequence of a result of [16] assuring the existence of continua i.e. closed and connected sets of periodic solutions for (P) assumed that the operator S: $\mathbb{R} \times L^p_{per}(\mathbb{R}; L^2(\mathcal{M})) \to L^p_{per}(\mathbb{R}; L^2(\mathcal{M}))$

$$u := S(\mu, w) \tag{9}$$

with u periodic solution of

$$u_t - \Delta_p u + R_e(x,u) = \mu Q(x,t)\beta(w) \quad \text{in } \mathcal{M} \times \mathbb{R}.$$

is relatively compact. Notice that $S(0, w) = u^0$ for any $w \in L^p_{per}(\mathbb{R}; L^2(\mathcal{M}))$. Now, we prove that S is relatively compact. For any bounded set $D \subset \mathbb{R} \times L^p_{per}(\mathbb{R}; L^2(\mathcal{M}))$ let $u \in S(D)$. Taking u as test function in the weak formulation, applying the Hölder and Young inequalities we get

$$\begin{aligned} &\frac{1}{2}\|u(T)\|^2_{L^2(\mathcal{M})} - \frac{1}{2}\|u(0)\|^2_{L^2(\mathcal{M})} \\ &\quad + \|\nabla u\|^p_{L^p(0,T;\, L^p(T\mathcal{M}))} + B\int_0^T\int_{\mathcal{M}} |u|^{r+1} dA dt \\ &\leq (\mu\|Q\|_\infty M + A)\int_0^T\int_{\mathcal{M}} |w|\, dA dt \\ &\leq (\mu\|Q\|_\infty M + A)^2 \frac{T|\mathcal{M}|}{2B} + \frac{B}{2}\int_0^T\int_{\mathcal{M}} |w|^2 dA dt. \end{aligned} \tag{10}$$

So,

$$\|u\|_{L^2(0,T;V)} \leq C. \tag{11}$$

Moreover, since the sequence $\{Q\beta(w_n)\}$ is weakly convergent in $L^1(0,T : L^2(\mathcal{M}))$, by the Corollary 2.3.2 of Vrabie [21], we deduce that there exists a sequence $u_n \in S(D)$ such that u_n strongly converges to u in $C_{per}(\mathbb{R}; L^2(\mathcal{M}))$ as n goes to infinity (and hence $u_n \to u$ in $L^p_{per}(\mathbb{R}; L^2(\mathcal{M}))$). To show that $S(\mu, w)$ is continuous, we consider u_n a periodic solution of the problem

$$(P_n) \quad u_{nt} - \Delta_p u_n + R_e(x, u_n) = \mu_n Q(x,t)\beta(w_n), \quad \text{in } \mathcal{M} \times \mathbb{R}$$

$0 < \mu_n < L$, $\mu_n \to \mu$ in $\mathbb{R}$. Taking u_n as test function in the weak formulation of solution to (P_n) and proceeding as above, we obtain

$$\|u_n\|_{L^p(0,T;V)} \leq C, \ \forall n \in \mathbb{N}, \tag{12}$$

and

$$\|(u_n)_t\|_{L^{p'}(0,T;V')} \leq C, \ \forall n \in \mathbb{N},$$

Hence, by Corollary 4 of Simon [18], there exists a subsequence denoted again by u_n such that $\forall T \geq 1$,

$$u_n \rightharpoonup u \quad \text{in } L^p(0,T;V), \ \forall T \geq 1 \tag{13}$$

$$u_n \to u \quad \text{in } L^p_{per}(\mathbb{R}; L^2(\mathcal{M})). \tag{14}$$

Since β is a maximal monotone graph, we have that

$$z_n = \beta(u_n) \rightharpoonup z \in \beta(u) \quad \text{in } L^2(0,T; L^2(\mathcal{M})), \ \forall T \geq 1 \tag{15}$$

moreover, as in [12],

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in } L^2(0,T; L^{p'}(T\mathcal{M})). \tag{16}$$

Passing to the limit as $n \to \infty$ in the weak formulation of solution to (P_n) we find that u is a weak bounded periodic solution to (P). So, by the Theorem 3.2 of [16] we get that the periodic solutions set of (P), contains a closed and connected containing $(0, u^0)$. The above conclusion remains true for the Budyko type model: it is enough to repeat the same estimates that we have done above, for a sequence β_n of bounded nondecreasing Lipschitz functions approximating β as n goes to infinity and passing to the limit, as in Arcoya, Diaz and Tello [1], by using that β is strongly-weakly closed since it is a maximal monotone graph as well as a topological result due to Whyburn [22].

The proof of the second part comes from the fact that if μ is large enough then any periodical solution of (P) is necessarily such that $u > -10$, so, $\beta(u) = \{M\}$ due to our assumption on β. Indeed, for μ large enough it is easy to construct a constant subsolution $\underline{u} > -10$. In that case the periodical solution of (P) is unique (see Remark 3) and the conclusion follows from the fact that the 1-periodic solutions of

$$u_t - \Delta_p u + R_e(x, u) = \mu Q(x, t) M \quad \text{in } \mathcal{M} \times \mathbb{R}.$$

depend monotonically and continuously of μ. ∎

ACKNOWLEDGMENTS

The authors thank the anonymous referees for their remarks leading to a more clear presentation of some parts of this paper.

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