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On the asymptotic behavior for a damped oscillator under a sublinear friction.

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Abstract. We show that there are two curves of initial data (x_0, v_0) for which the solutions $x(t)$ of the corresponding Cauchy problem associated to the equation $x_{tt} + |x_t|^{\alpha-1} x_t + x = 0$, where $\alpha \in (0,1)$, vanish after a finite time. By using asymptotic and methods and comparison arguments we show that for many other initial data the solutions decay to 0, in an infinite time, as $t^{-\alpha/(1-\alpha)}$.

Sobre el comportamiento asintótico de un oscilador amortiguado con un término de fricción sublineal

Resumen. Mostramos la existencia de dos curvas de datos iniciales (x_0, v_0) para las cuales las soluciones $x(t)$ correspondientes del problema de Cauchy asociado a la ecuación $x_{tt} + |x_t|^{\alpha-1} x_t + x = 0$, supuesto $\alpha \in (0,1)$, se anulan identicamente despues de un tiempo finito. Mediante métodos asintóticos y argumentos de comparación mostramos que para muchos otros datos iniciales las soluciones decaen a 0, en un tiempo infinito, como $t^{-\alpha/(1-\alpha)}$.

1. Introduction

We study the asymptotic behavior of solutions of the equation

$$mx_{tt} + \mu |x_t|^{\alpha-1} x_t + kx = 0 \tag{1}$$

where $\alpha \in (0,1)$ and $\mu, k > 0$. We shall work with the formulation

$$x_{tt} + |x_t|^{\alpha-1} x_t + x = 0 \tag{2}$$

which is attaint by dividing by k and by introducing the rescaling $\widetilde{x}(\widetilde{t}) = \beta^{1/(\alpha-1)} x(\lambda \widetilde{t})$ where $\lambda = \frac{\sqrt{m}}{\sqrt{k}}$ and $\beta = \frac{\mu}{k^{(2-\alpha)/2} m^{\alpha/2}}$. Notice that the x-rescaling fails for $\alpha = 1$. In that case there is no well defined scale for x and the equation is reduced to $x_{tt} + \beta x_t + x = 0$ with $\beta = \frac{\mu}{\sqrt{km}}$ remaining as a parameter to characterize the dynamics. The limit case $\alpha \to 0$ corresponds to the Coulomb friction equation

$$x_{tt} + sign(x_t) + x \ni 0 \tag{3}$$

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where $sign$ is the maximal monotone graph of $\mathbb{R}^2$ given by $sign(r) = -1$, if $r < 0, [-1, 1]$ if $r = 0$, and 1 if $r > 0$. The limit equation when $\alpha \to 1$ corresponds with the linear damping equation

$$x_{tt} + x_t + x = 0. \tag{4}$$

We recall that, even if the nonlinear term $|x_t|^{\alpha-1} x_t$ is not a Lipschitz continuous function of x_t , the existence and uniqueness of solutions of the associate Cauchy problem

$$P_\alpha \left\{ \begin{array}{l} x_{tt} + |x_t|^{\alpha-1} x_t + x = 0 \quad t > 0, \\ x(0) = x_0, \; x_t(0) = v_0 \end{array} \right.$$

(and of the limit problems P_0 and P_1 corresponding to the equations (3) and (4) respectively) is well known in the literature: see, e.g. Brezis [1]. An easy application of the results of the above reference yields to a rigorous proof of the convergence of solutions when $\alpha \to 0$ and $\alpha \to 1$.

The asymptotic behavior, for $t \to \infty$, of solutions of the limit problems P_0 and P_1 is well known (see, for instance, Jordan and Smith [5]). In the first case the decay is exponential. In the second one it is easy to see that "given x_0 and v_0 there exist a finite time $T = T(x_0, v_0)$ and a number $\zeta \in [-1, 1]$ such that $x(t) \equiv \zeta$ for any $t \geq T(x_0, v_0)$". For problem P_α it is well-known that $(x(t), x_t(t)) \to (0,0)$ as $t \to \infty$ (see, e.g. Haraux [4]). For a numerical study of (P_α) see [6].

The main result of this paper is to show that the generic asymptotic behavior above described for the limit case P_0 is only exceptional for the sublinear case $\alpha \in (0, 1)$ since the generic orbits $(x(t), x_t(t))$ decay to $(0, 0)$ in a infinite time and only two uniparametric families of them decay to $(0, 0)$ in a finite time: in other words, when $\alpha \to 0$ the exceptional behavior becomes generic.

2. Formal results via asymptotic arguments

We can rewrite the equation (2) in as the planar system

$$\left\{ \begin{array}{l} x_t = y \\ y_t = -x - |y|^{\alpha-1} y \end{array} \right. \tag{5}$$

which, by eliminating the time variable, for $y \neq 0$, leads to the differential equation of the orbits in the phase plane

$$y_x = \frac{-x - |y|^{\alpha-1} y}{y} \tag{6}$$

and that allows us to carry out a phase plane description of the dynamics.

We remark that the plane phase is antisymmetric since if $y = \varphi(x)$ is a solution of (6) then the function $y = -\varphi(-x)$ is also solution. So, it is enough to describe a semiplane (for instance $x \geq 0$). By multiplying by x and y, respectively, we get that $(x^2 + y^2)_t = 2|y|^{\alpha+1}$. On the other hand, it is easy to see that $(1/x, 1/y)$ satisfy a system which has the point $(0, 0)$ as a spiral unstable critical point. For values of $x^2 + y^2 >> 1$ the orbits of the system are given, in first approximation, by $x^2 + y^2 = C$ because $|y|^{\alpha-1} y$ is small compared with x. The effect of this term is to decrease slowly C with time giving the trajectory a spiral character. For $\alpha = 1$ the character of the trayectories close to the origin depends on the parameter β. For $\beta > \beta_c := 2$ the origin is a stable mode and for $\beta < \beta_c$ is a stable spiral corresponding to underdamped oscillations. It should be noticed that for $\alpha > 1$ the origin becomes a stable spiral point. The limit case $\alpha \to +\infty$ can be described analytically with two-time scale methods (see [3]).

We shall prove that there are two modes of approach to the origin and so that the origin $(0, 0)$ is a node for the system (5). The lines of zero slope are given by

$$-x = |y|^{\alpha-1} y. \tag{7}$$

So the convergence to $(0,0)$ is only possible through the regions $\{(x,y) : x > 0, y < -x^{1/\alpha}\} \cup \{(x,y) : x < 0, y > (-x)^{1/\alpha}\}$. Let us see that the "ordinary" mode corresponds to orbits that are very close to the ones corresponding to small effects of the inertia. Due to the symmetry it is enough to describe this behavior for the orbits approaching the origin with values of $x > 0$ and $y < 0$. Let $-y = \widetilde{y} > 0$. Equation (6) takes the form

$$\widetilde{y}\widetilde{y}_x = -x + \widetilde{y}^\alpha. \tag{8}$$

The line of zero slope is $\widetilde{y} = x^{1/\alpha}$ and we search for orbits obeying, for $0 < x << 1$, to the expression $\widetilde{y} = x^{1/\alpha} + z(x)$ for some function $z(x)$. If we anticipate the condition $0 < z(x) << x^{1/\alpha}$, equation (6) takes the "linearized form" $\frac{1}{\alpha}x^{(\frac{1}{\alpha}-1)}z + x^{\frac{1}{\alpha}}z_x - \alpha x^{(1-\frac{1}{\alpha})}z = 0$. Thus the first term can be neglected, compared with the last one, and then the solution can be written as $z(x) \sim C\exp\{-[\alpha^2/2(1-\alpha)]x^{-\frac{2(1-\alpha)}{\alpha}}\}$ with C an arbitrary constant (which explain the name of "ordinary" orbits). This type of orbits are given, close to the origin, by the approximate equation (7), which for the orbits that reach the origin from below implies that $\widetilde{y} \sim x^{1/\alpha} \sim -\frac{dx}{dt}$ and so, integrating the simplified equation

$$\frac{dx}{dt} = -x^{1/\alpha} \tag{9}$$

we get that

$$x(t) \sim [\frac{\alpha}{(1-\alpha)(t+t_1)}]^{\alpha/(1-\alpha)} \tag{10}$$

and so that it takes an infinite time to reach the origin.

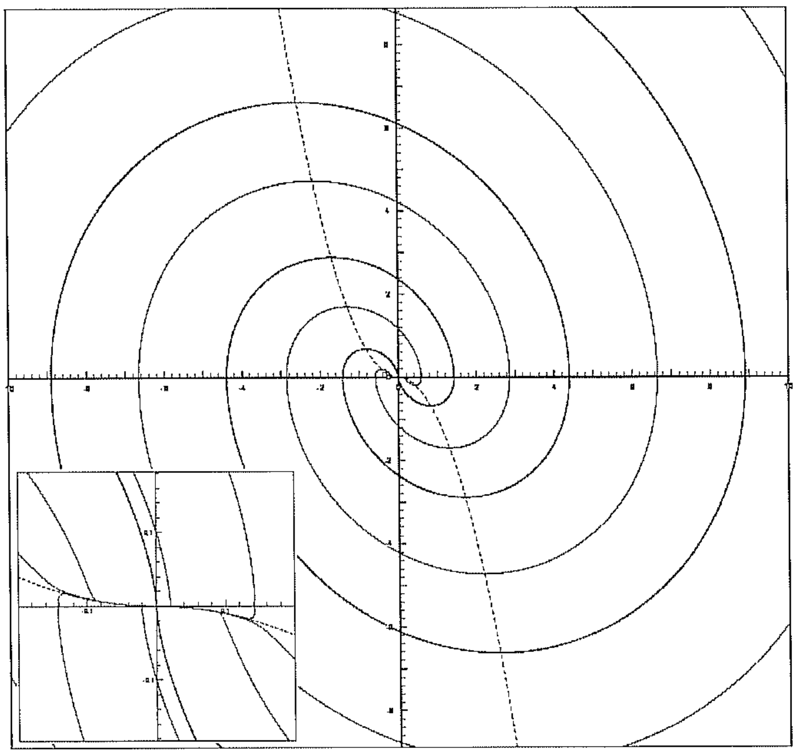

Figure 1. The two exceptional orbits and, in the small figure, several ordinary orbits entering to the origin tangentially to the zero slop curve for $\alpha = 0.5$

Some different orbits approaching the origin can be found by searching among solutions with large values of $|y|$ compared with $|x|^{1/\alpha}$. Thus, close to the origin, the orbits with negative y are "very close" to the solutions of the equation found by replacing (8) by the simplified the equation

$$\widetilde{y}\widetilde{y}_x = \widetilde{y}^\alpha \tag{11}$$

corresponding to a balance of inertia and damping. The solution ending at the origin ($\widetilde{y}(0) = 0$) is given by

$$\widetilde{y}(x) = -\{(2-\alpha)x\}^{1/(2-\alpha)}. \tag{12}$$

Notice that it involves no arbitrary constant. So this curve is unique (a symmetric curve arises for $y > 0$ and $x < 0$) which justifies the term of "extraordinary" orbit. The time evolution of this orbit is given, for $x << 1$, by integrating the equation

$$-\frac{dx}{dt} = [(2-\alpha)x]^{1/(2-\alpha)} \tag{13}$$

and so $x(t) = \frac{1}{(2-\alpha)}[\frac{(2-\alpha)(1-\alpha)}{2\alpha}(t_0 - t)_+]^{(2-\alpha)/(1-\alpha)}$, where in general $h(t)_+ = \max\{0, h(t)\}$. This indicate that the motion (of this approximated solution) ends at a finite time, t_0, determined by the initial conditions which, by (13) must satisfy that $v_0 \sim \pm[(2-\alpha)\,|x_0|]^{1/(2-\alpha)}$ We point out that the two exceptional orbits emanating from the origin spiral around the origin when $x^2 + y^2$ grows toward infinity and so each of them is a separatrix curve in the phase plane. Notice that due to the autonomus nature of the equation, if $x(t)$ is the solution of the Cauchy problem (P_α) of initial data (x_0, v_0) then for any parameter $\tau \geq 0$ the function $\widetilde{x}(t) := x(t+\tau)$ coincides with the solution of (P_α) of initial data $(x(\tau), x_t(\tau))$. In this way, the above extraordinary orbits give rise to two curves of initial data for which the corresponding solutions of (P_α) vanish after a finite time.

We end this section by pointing out that the solution of problem (P_α) for $0 < \alpha << 1$ takes an asymptotic form which can be easily described. The differential equations of the orbits "simplify" if $y \neq 0$ is finite and $\alpha \to 0$ to $yy_x = -x - 1$ for $y > 0$ and $\widetilde{y}\widetilde{y}_x = -x + 1$ for $\widetilde{y} = -y > 0$. The solutions are circles with center at $x = -1$ if $y > 0$ and center $x = 1$ if $y < 0$ joined. An orbit formed with half circles with centers at $x = -1$ or $x = 1$ when it hits the interval $(0, 1)$ from below it is transformed into an orbit that reaches the origin following very closely that segment, governed by the equation (9) of solution (10). In the limit $\alpha \to 0$ we found that any point $\zeta \in [-1, 1]$ is an asymptotically stable stationary state of (P_0).

3. Estimates on the decay

In a previous work, by using a fixed point argument, we show

Theorem 1 ([2]) *There exists two curves Γ_+ and Γ_- of initial data (x_0, v_0) for which the solutions $x(t)$ of the corresponding Cauchy problem (P_α) vanish after a finite time.*

It is possible to give some additional results on these two curves:

Theorem 2 *(i) Near the origin the curves Γ_+ and Γ_- can be represented by two functions, $y = \varphi_+(x)$ and $y = \varphi_-(x)$, solutions of the equation (6), where $\varphi_+ : [0, \varepsilon] \to (-\infty, 0]$ and $\varphi_- : [-\varepsilon, 0] \to [0, +\infty)$, for some $\varepsilon > 0$.*
(ii) Functions φ_+ and φ_-, satisfy that $\varphi_\pm(0) = 0$

$$-\infty < \int_0^\varepsilon \frac{ds}{\varphi_+(s)} \text{ and } \int_{-\varepsilon}^0 \frac{ds}{\varphi_-(s)} < +\infty. \tag{14}$$

In particular, $\varphi'_+(x) \downarrow -\infty$ when $x \downarrow 0$ and *$\varphi'_-(x) \uparrow +\infty$ when $x \uparrow 0$.*

(iii) We have

$$-Cx^{\frac{1}{2-\alpha}} \leq \varphi_+(x) \leq -x^{\frac{1}{\alpha}} \textit{ for } x \in [0,\varepsilon] \textit{ and}$$
$$(-x)^{\frac{1}{\alpha}} \leq \varphi_-(x) \leq C(-x)^{\frac{1}{2-\alpha}} \textit{ for } x \in [-\varepsilon, 0],$$

for some $C > 0$.
(iv) There exists a $x_s \in (0,\varepsilon]$ *such that* $\varphi_+(x_s) = -(x_s)^{\frac{1}{\alpha}}$ *and* $(-x_s)^{\frac{1}{\alpha}} = \varphi_-(x_s)$. *Moreover the regions* $D_+ := \{(x,y)\colon x \in [0,x_s]$ *and* $\varphi_+(x) \leq y \leq -x^{\frac{1}{\alpha}}\}$, $D_- := \{(x,y)\colon x \in [-x_s, 0]$ *and* $(-x)^{\frac{1}{\alpha}} \leq y \leq \varphi_-(x)\}$ *are time invariants for equation (6).*

In order to prove that the decay to zero in an infinite time is more generic than the decay to zero in a finite time we need to obtain sharper invariants regions

Theorem 3 *(i) There exists a* $\delta \in (0, x_s)$ *small enough such that the regions* $D_+^{\delta} := \{(x,y) \in D_+ : x \in [0, x_s - \delta]$ *and* $-x^{\frac{1}{\alpha}} - \delta \exp\{-[\alpha^2/2(1-\alpha)]x^{-\frac{2(1-\alpha)}{\alpha}}\} < y \leq -x^{\frac{1}{\alpha}}\}$, $D_-^{\delta} := \{(x,y) \in D_- : x \in [-x_s + \delta, 0]$ *and* $(-x)^{\frac{1}{\alpha}} \leq y \leq (-x)^{\frac{1}{\alpha}} + \delta \exp\{[\alpha^2/2(1-\alpha)]x^{-\frac{2(1-\alpha)}{\alpha}}\})\}$ *are time invariants for equation (6).*
(ii) If $(x_0, v_0) \in D_+^{\delta}$ *(respectively* D_-^{δ}*) then the solution* $x(t)$ *of* (P_α) *satisfies that* $x(t) \geq Ct^{-\alpha/(1-\alpha)}$ *(respectively* $x(t) \leq -Ct^{-\alpha/(1-\alpha)}$*) for some* $C > 0$ *and any* $t > 0$.

PROOF OF THEOREM 2. Since the solutions of equation (6) converge to the origin with zero or $\pm\infty$ slope the representation of curves Γ_+ and Γ_- by $y = \varphi_+(x)$ and $y = \varphi_-(x)$, is clear once we assume $|x|$ small enough. Then for $t \geq t_0$, with t_0 large enough the solutions $x(t)$ of the corresponding Cauchy problem (P_α) vanish after a finite time.and satisfy the first order ordinary differential equation $x_t(t) = \varphi_+(x(t))$ and $x_t(t) = \varphi_+(x(t))$ respectively. Therefore the necessary conditions (14) holds by (easy) well known results. To prove (iii) we call (as in the previous Section) $-y = \widetilde{y}$. So, for instance, $\widetilde{y} = \widetilde{\varphi}_+(x) > 0$, with $\widetilde{\varphi}_+ = -\varphi_+$ is a solution of the new orbits equation $\widetilde{y}\widetilde{y}_x - \widetilde{y}^{\alpha} = -x$. It is easy to see that this equation admits some explicit sub and supersolutions of the form $x^{\frac{1}{\alpha}}$ and $Cx^{\frac{1}{2-\alpha}}$, respectively, and so the result holds by a comparison argument. Finally, the existence of $x_s \in (0,\varepsilon)$ assured in (iv) can be obtained, for instance, by using the inequalities of (iii) and equation (6). The direction of the flow at the boundaries of D_+ and D_- implies the time invariance of these regions. ■

PROOF OF THEOREM 3. Let $(x_0, v_0) \in D_+^{\delta}$. let us construct an auxiliary function satisfying that $\widetilde{Y}\widetilde{Y}_x - \widetilde{Y}^{\alpha} \leq -x$ on $(0, x_0)$ and $\widetilde{Y}(x_0) \geq v_0$. Then, by comparison arguments (forward in the x-direction) we get that necesarily $\widetilde{\varphi}_+(x) \leq \widetilde{Y}(x)$ on $(0, x_0)$. Such a function $\widetilde{Y}(x)$ can be constructed in the form $\widetilde{Y}(x) = ax^{1/\alpha} + z(x)$ with $z(x) = C\exp\{-[\alpha^2/2(1-\alpha)]x^{-\frac{2(1-\alpha)}{\alpha}}\}$ with $C > 0$ arbitrary but small enough (use the fact that $\frac{a^2}{\alpha}x^{\frac{2-\alpha}{\alpha}} + x - \frac{1}{2}a^{\alpha}x < 0$ for $x \in (0, x_s)$ if $a > 0$ is suitably chosen and that $z(x) < ax^{1/\alpha}$). Finally, for initial data in D_+^{δ} we get that $x_t(t) \geq -2x(t)^{\frac{1}{\alpha}}$ for $t \geq 0$ and the decay inequality follows from the comparison with the exact solution of $x_t(t) = -2x(t)^{\frac{1}{\alpha}}$. ■

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