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# Spatial and continuous dependence estimates in linear viscoelasticity

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## Abstract

In this paper we consider the problem determined by the anti-plane shear dynamic deformations for the linear theory of viscoelasticity. First, we prove existence of solutions of the problem determined in a semi-infinite strip. Then, we show that the rate of decay of the end effects in this problem is faster than that known for the Laplace equation. In the last section, we study the influence of the mass density on the decay of end effects.

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## 1. Introduction

Some materials display a behaviour with a scope wider than that of a simply elastic or purely viscous medium. A relevant type of these media are the viscoelastic solids, which exhibit a combined behaviour of elastic and viscous materials. For background material on these questions the reader is referred to [8,13].

Specifically, we consider the linearized anti-plane dynamic deformation of a semi-infinite viscoelastic homogeneous isotropic medium of cross-section  $R = \{(x_1, x_2): x_1 > 0, x_2 \in (0, L)\}$ . The displacement field is assumed of the form  $(0, 0, u(x_1, x_2, t))$  (as mentioned in Horgan [6], anti-plane shear may be regarded

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as a natural complement to that of plane strain, were the displacement field has the form  $(u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0)$ . The momentum conservation leads to the equation

$$u_{tt} = \gamma \Delta u_t + \mu \Delta u, \quad (1.1)$$

where we have assumed that the material has a positive constant density (which without loss of generality can be taken as 1), the positive constant  $\mu$  is the Lamé constant and the positive constant  $\gamma$  measures the viscoelastic damping. In that model the stress depends on the strain ( $\epsilon = \nabla u$ ) and on its time velocity  $\epsilon_t$  through a linear constitutive relation of the form

$$\mu \nabla u + \gamma \nabla u_t.$$

For more information on linear viscoelasticity we refer to the review article of Leitman and Fisher [8]. A more recent general reference dealing with nonlinear materials and memory effects is the book [13] (see also some results on the propagation of disturbances in [2]). Concerning the existence of solutions see [4] and [14]. We also point out the decay results (of Phragmen–Lindelöf type) obtained in [12].

It is worth remarking that there are no previous contributions concerning existence of solutions for the problem determined by (1.1) with Dirichlet boundary conditions. Thus, our first aim is to guarantee the existence of solutions for Eq. (1.1) in a semi-infinite strip. One of the main goals of this paper is to prove that the solutions of (1.1) decay spatially (for any fixed time) faster than the solutions of the associated equilibrium equation. A similar result was proved in Horgan et al. [7] for the case of the parabolic heat equation. We shall prove (see the estimates (3.21) and (3.35)) that this property also holds for the parabolic equation (1.1). In order to do that we adapt some arguments developed in [9] for the case of elliptic equations.

The last section of the paper is devoted to the study of the continuous dependence with respect to the mass density  $\rho$ . More precisely, we consider a similar material but with a different mass density  $\rho$  and the associated anti-plane shear displacement  $\bar{u}$ , i.e., satisfying

$$\rho \bar{u}_{tt} = \gamma \Delta \bar{u}_t + \mu \Delta \bar{u}. \quad (1.2)$$

We obtain an estimate on

$$\int_0^t \int_{x_1}^{\infty} \int_0^L |u - \bar{u}|^2 ds d\tau,$$

for any  $t > 0$  and  $x_1 \geq 0$  in terms of  $(\rho - 1)^2$  and other multiplicative terms.

Summation and differentiation conventions will be used throughout this paper. We recall that summation over repeated indices is implied and the suffix “ $i$ ” denotes  $\partial/\partial x_i$ .

## 2. Existence of solutions

In this section we study the existence of solutions of the boundary value problem determined by

$$u_{tt} = \gamma \Delta u_t + \mu \Delta u + f(x_1, x_2, t), \quad \text{in } R \times (0, \infty), \tag{2.1}$$

$$\begin{aligned} u(x_1, 0, t) = u(x_1, L, t) = u(0, x_2, t) = 0, \\ x_1 > 0, \quad x_2 \in (0, L), \quad t > 0, \end{aligned} \tag{2.2}$$

$$u(x_1, x_2, 0) = u_0(x_1, x_2), \quad u_t(x_1, x_2, 0) = v_0(x_1, x_2), \quad \text{in } R. \tag{2.3}$$

Here  $\mu$  and  $\nu$  are two positive constants and  $f(x_1, x_2, t)$ ,  $u_0(x_1, x_2)$  and  $v_0(x_1, x_2)$  are given functions.

We need to assume some conditions on the behaviour of  $f$ ,  $u_0$ ,  $v_0$ . Given  $\alpha \geq 0$ , we denote by  $L^2_\alpha(R)$ , the class of functions  $u$  such that  $\exp(-\alpha x_1)u \in L^2(R)$ . In a similar way, we denote by  $W^{p,q}_\alpha(R)$  (respectively,  $W^{p,q}_{0,\alpha}(R)$ ) the class of functions  $u$  such that  $\exp(-\alpha x_1)u \in W^{p,q}(R)$  (respectively,  $W^{p,q}_0(R)$ ). In the special case that  $q = 2$ , we shall use the notation  $H^p_\alpha(R) = W^{p,2}_\alpha(R)$  (respectively,  $H^p_{0,\alpha}(R) = W^{p,2}_{0,\alpha}(R)$ ). The inner product in  $L^2_\alpha(R)$  is defined by

$$\langle u, v \rangle = \frac{1}{2} \int_R \exp(-2\alpha x_1) uv \, da. \tag{2.4}$$

Here and in what follows  $da = dx_1 dx_2$ . The effect of the function  $\exp(-2\alpha x_1)$  is to allow solutions growing at infinite. Some other uses of the weighted functions are due to Galdi and Rionero [5] and Straughan [16].

If we denote by  $v = u_t$ , the problem (2.1)–(2.3) can be written as

$$\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{A} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad (u(0), v(0)) = (u_0, v_0), \tag{2.5}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & -I \\ -\mu \Delta & -\gamma \Delta \end{pmatrix}. \tag{2.6}$$

If we denote by

$$\mathbf{Z}_\alpha = H^1_{0,\alpha}(R) \times L^2_\alpha(R),$$

we may define the inner product

$$\langle (u, v), (\tilde{u}, \tilde{v}) \rangle = \frac{1}{2} \int_R \exp(-2\alpha x_1) (\mu u_j \tilde{u}_j + v \tilde{v}) \, da, \tag{2.7}$$

and the norm

$$\| (u, v) \|^2 = \frac{1}{2} \int_R \exp(-2\alpha x_1) (\mu u_j u_j + v^2) \, da.$$

The domain of the operator  $\mathbf{A}$  contains the set  $[H_\alpha^2(R) \cap H_{0,\alpha}^1(R)]^2$  which is dense in  $Z_\alpha$ . We have the following result:

**Theorem 1.** Assume that  $f \in L^2(0, T; L_\alpha^2(R))$ ,  $u_0 \in H_\alpha^1(R)$  and  $v_0 \in L_\alpha^2(R)$ . Then, there exists a unique mild solution  $(u(t), v(t))$  of the abstract Cauchy problem determined by (2.5).

**Proof.** By virtue of Theorem 3.17 of [3, p. 105], it is enough to prove the existence of a positive constant  $\nu$  such that the operator  $\overline{\mathbf{A}}, \nu = \mathbf{A} + \nu \mathbf{I}$  is maximal monotone. First, we prove that there exists a positive constant  $\nu_1$  such that

$$\langle \mathbf{A}(u, v), (u, v) \rangle \geq -\nu_1 \|(u, v)\|^2, \quad (2.8)$$

for all  $(u, v)$  at the domain of  $\mathbf{A}$ . Integrate by parts to obtain

$$\langle \mathbf{A}(u, v), (u, v) \rangle = \int_R \exp(-2\alpha x_1) (\gamma v_j v_j - 2\alpha \mu v u_1 - 2\alpha \gamma v v_1) da.$$

Use Schwarz's inequality and the arithmetic–geometric mean inequality to obtain

$$2\alpha \mu v u_1 \leq \frac{2\alpha^2 \mu v^2}{\epsilon_1} + \frac{\mu \epsilon_1}{2} u_1^2, \quad 2\alpha \gamma v v_1 \leq \frac{2\alpha^2 \gamma v^2}{\epsilon_2} + \frac{\gamma \epsilon_2}{2} v_1^2,$$

where  $\epsilon_1$  and  $\epsilon_2$  are two arbitrary positive constants. If we take (for instance)  $\epsilon_2 = 2$ , we may select

$$\nu_1 = \max \left( \epsilon_1, \frac{\alpha^2 \mu}{\epsilon_1} + \alpha^2 \gamma \right).$$

Second, we prove that there exists a positive constants  $\nu$  such that

$$\text{Rang}(\nu \mathbf{I} + \mathbf{A}) \supset (H_\alpha^2(R))^2.$$

Let  $(u^*, v^*) \in (H_\alpha^2(R))^2$ . We must prove that the system

$$\nu u - v = u^*, \quad \nu v - \mu \Delta u - \gamma \Delta v = v^* \quad (2.9)$$

has a solution  $(u, v) \in D(\mathbf{A})$ . If we substitute the first equation into the second, we have

$$\nu^2 u - \mu \Delta u - \gamma \nu \Delta u = v^* + \nu u^* - \gamma \Delta u^*, \quad (2.10)$$

which has a unique solution. As  $u \in H_\alpha^2(R) \cap H_{0,\alpha}^1(R)$ , we see that  $v \in H_{0,\alpha}^1(R)$ . Lumer–Phillips corollary to the Hille–Yosida theorem leads to the theorem.  $\square$

Since the operator  $\overline{\mathbf{A} + \nu \mathbf{I}}$  is maximal monotone, we obtain some estimates for the solutions (see [3, Theorem 3.17]). Thus, the dynamic problem of the anti-plane shear deformations in viscoelasticity is well posed.

It is worth remarking that if  $(u_0, v_0) \in D(\mathbf{A})$  and  $f$  is a function of bounded variation in  $L_\alpha^2(R)$ , then the time derivative of the solution belongs to  $L^\infty(0, T; Z_\alpha)$  (see [3, Theorem 3.17]).

### 3. The spatial decay is faster than in the stationary case

In this section we investigate the spatial behaviour of the solutions of the problem

$$\mu \Delta u + \gamma \Delta u_t = u_{tt}, \quad \text{in } R \times [0, \infty), \tag{3.1}$$

$$u(x_1, 0, t) = u(x_1, L, t) = 0, \quad x_1 > 0, \quad t > 0, \tag{3.2}$$

$$\int_0^\infty \exp(-2\alpha x_1) u^2(x_1, x_2, t) dx_1 < \infty, \tag{3.3}$$

$$u(0, x_2, t) = g(x_2, t), \quad 0 \leq x_2 \leq L, \quad t > 0, \tag{3.4}$$

$$u(x_1, x_2, 0) = u_t(x_1, x_2, 0) = 0, \quad \text{in } R. \tag{3.5}$$

Here  $2\alpha$  is a positive constant which is less than  $\omega$  (see (3.13)) and  $g(x_2, t)$  is such that satisfies the condition:

(Hg) There exist  $\alpha > 0$  and  $G(t)(x_1, x_2) = G(x_1, x_2, t) \in H^2(0, T; L^2_\alpha(R)) \cap L^2(0, T; H^2_\alpha \cap H^1_{\alpha, \Gamma_2}(R))$  for all  $T > 0$  such that  $G_t \in L^2(0, T; H^1_\alpha(R))$  and  $G(0, x_2, t) = g(x_2, t)$  a.e.  $t \in (0, T)$ .

Here the space  $H^1_{\alpha, \Gamma_2}(R)$  is the adherence in  $H^1_\alpha(R)$  of the  $C^\infty$ -functions which vanish when  $x_2 = 0, L$ .

Now, we obtain the existence result of the solutions of the problem determined by (3.1)–(3.5). It is a consequence of Theorem 1.

**Theorem 2.** *Assume that (Hg) is satisfied. Then, there exists a unique mild solution  $(u(t), v(t))$  of the problem determined by (3.1)–(3.5).*

**Proof.** Let  $w = u - G$ . Then

$$w_{tt} - \gamma \Delta w_t - \mu \Delta w = f(x_1, x_2, t), \tag{3.6}$$

with

$$f(x_1, x_2, t) = G_{tt} - \gamma \Delta G_t - \mu \Delta G. \tag{3.7}$$

So, from (Hg),  $f \in L^2(0, T; L^2_\alpha(R))$ . Moreover,

$$w(0, x_2, t) = g(x_2, t) - G(0, x_2, t) = 0, \tag{3.8}$$

$$w(x_1, 0, t) = -G(x_1, 0, t) = 0,$$

$$w(x_1, L, t) = -G(x_1, L, t) = 0, \tag{3.9}$$

and

$$w(x_1, x_2, 0) = -G(x_1, x_2, 0) = u_0(x_1, x_2),$$

$$w_t(x_1, x_2, 0) = -G_t(x_1, x_2, 0) = v_0(x_1, x_2). \tag{3.10}$$

From Theorem 1, we derive the existence of a solution  $w$  which satisfies conditions (3.8)–(3.10). The theorem is proved by taking  $u = w + G$ .  $\square$

Having proved the existence of solutions we turn to the study of their spatial behaviour. We denote  $\lambda = \mu\gamma^{-1}$ . For  $\beta \geq 4\lambda$  we define the function

$$H_\beta(z, t) = \frac{1}{2} \int_0^t \int_0^L \exp((2\lambda - \beta)\tau) (u_\tau + \lambda u)^2 dl d\tau, \quad (3.11)$$

and we have the following result:

**Theorem 3.** *Let  $u$  be the mild solution of the initial boundary value problem (3.1)–(3.5). Then, the function  $H_\beta$  satisfies the condition*

$$H_\beta(z, t) \leq K_{g,\beta}(t) \exp(-\omega z), \quad \text{for all } z \geq 0, \quad (3.12)$$

where

$$\omega = \sqrt{2 \frac{\pi^2}{L^2} + \left( \frac{\beta}{\gamma} - \frac{4\mu}{\gamma^2} \right)} \quad (3.13)$$

and

$$K_{g,\beta}(t) = \frac{1}{2} \int_0^t \int_0^L \exp((2\lambda - \beta)\tau) (g_\tau + \lambda g)^2 dl d\tau. \quad (3.14)$$

**Proof.** Let us start with a function  $g$  smooth enough (in such a way that its corresponding functions  $f$ ,  $u_0$  and  $v_0$  are regular enough). Introducing the function

$$v(x_1, x_2, t) = \exp(\lambda t) u(x_1, x_2, t), \quad (3.15)$$

we have

$$H_\beta(z, t) = \frac{1}{2} \int_0^t \int_0^L \exp(-\beta\tau) (v_\tau)^2 dl d\tau. \quad (3.16)$$

Direct differentiation shows

$$\frac{\partial H_\beta}{\partial z} = \int_0^t \int_0^L \exp(-\beta\tau) v_\tau v_{\tau 1} dl d\tau. \quad (3.17)$$

Using the evolution equation, the boundary conditions (3.2) and the divergence theorem we have

$$\begin{aligned} \frac{\partial^2 H_\beta}{\partial z^2} &= \int_0^t \int_0^L \exp(-\beta\tau) \left( v_{\tau i} v_{\tau i} + \left( \frac{\beta}{2\gamma} - 2\mu\gamma^{-2} \right) (v_\tau)^2 \right) dl d\tau \\ &+ \frac{1}{2\gamma} \int_0^L \exp(-\beta t) ((v_t)^2 + \mu^2\gamma^{-2}v^2) dl. \end{aligned} \tag{3.18}$$

In the next step of the proof we use the Poincaré inequality. Thus, we recall that the estimate

$$\int_0^L u^2 d\tau \leq \left( \frac{2L}{\pi} \right)^2 \int_0^L (u_2)^2 dx_2, \tag{3.19}$$

is satisfied for functions such that  $u(0) = 0$ .

Equality (3.18) and inequality (3.19) imply the estimate

$$\begin{aligned} \frac{\partial^2 H_\beta}{\partial z^2} &\geq 2 \left( \frac{\pi^2}{L^2} + \left( \frac{\beta}{2\gamma} - \frac{2\mu}{\gamma^2} \right) \right) H_\beta + \int_0^t \int_0^L \exp(-\beta\tau) v_{\tau 1} v_{\tau 1} dl d\tau \\ &+ \frac{1}{2\gamma} \int_0^L \exp(-\beta t) (v_t)^2 dl. \end{aligned} \tag{3.20}$$

The estimate (3.12) is a direct consequence of estimate (3.20) and the assumption that  $2\alpha < \omega$  (see [9]).

In the general case we may approximate the solution by a sequence of solutions satisfying the regularity conditions assumed at the beginning of the proof. If we pass to the limit and use the fact that (3.12) depends merely on  $K_{g,\beta}(t)$  and  $\omega$ , we obtain the desired result.  $\square$

**Theorem 4.** *Let  $u$  be the mild solution of the initial boundary value problem (3.1)–(3.5). Then, the function  $H_\beta$  satisfies*

$$\begin{aligned} H_\beta(z, t) &\leq \exp(-2\delta t) \\ &\times \left( \left( \frac{1}{4\pi\gamma} \right)^{1/2} z \int_0^t \frac{\exp(\delta\tau) K_{g,\beta}^{1/2}(\tau)}{(t-\tau)^{3/2}} \exp\left(-\frac{z^2}{4\gamma(t-\tau)}\right) d\tau \right)^2, \end{aligned} \tag{3.21}$$

where

$$\delta = \frac{\gamma\omega^2}{2}. \tag{3.22}$$

**Proof.** We use a comparison argument to improve the decay rate. As in Theorem 3, we first assume that  $g$  is smooth enough in the sense that the corresponding  $u_0, v_0$  and  $f$  are regular. From (3.20) we see that

$$\begin{aligned}
 H_\beta \frac{\partial^2 H_\beta}{\partial z^2} - \frac{1}{2} \left( \frac{\partial H_\beta}{\partial z} \right)^2 &\geq \omega^2 H_\beta^2 + \gamma^{-1} H_\beta \frac{\partial H_\beta}{\partial t} \\
 + \frac{1}{2} \int_0^t \int_0^L \exp(-\beta\tau) (v_\tau)^2 \, dl \, d\tau &\int_0^t \int_0^L \exp(-\beta\tau) (v_{\tau 1})^2 \, dl \, d\tau \\
 - \frac{1}{2} \left( \int_0^t \int_0^L \exp(-\beta\tau) v_\tau v_{\tau 1} \, dl \, d\tau \right)^2 &.
 \end{aligned} \tag{3.23}$$

If we denote

$$P_\beta(z, t) = (H_\beta(z, t))^{1/2}, \tag{3.24}$$

the Schwarz inequality leads to

$$\frac{\partial^2 P_\beta}{\partial z^2} \geq \frac{\omega^2}{2} P_\beta + \gamma^{-1} \frac{\partial P_\beta}{\partial t}. \tag{3.25}$$

Thus, the function  $P_\beta(z, t)$  satisfies the problem determined by (3.25), the initial condition

$$P_\beta(z, 0) = 0, \quad \text{for all } z \geq 0, \tag{3.26}$$

the boundary condition

$$P_\beta(0, t) = \left( \frac{1}{2} \int_0^t \int_0^L \exp(-\beta\tau) \left( \frac{d}{d\tau} [\exp(\lambda\tau) g^2(x_2, \tau)] \right)^2 \, dl \, d\tau \right)^{1/2}, \tag{3.27}$$

and from Theorem 3, we have

$$P_\beta(z, t) \rightarrow 0 \quad \text{uniformly in } t \text{ as } z \rightarrow \infty. \tag{3.28}$$

Set

$$Q_\beta(z, t) = \exp(\delta t) P_\beta(z, t). \tag{3.29}$$

The function  $Q_\beta$  satisfies

$$\begin{aligned}
 \frac{\partial^2 Q_\beta}{\partial z^2} &\geq \gamma^{-1} \frac{\partial Q_\beta}{\partial t}, \\
 Q_\beta(z, 0) &= 0, \quad \text{for all } z \geq 0, \\
 Q_\beta(0, t) &= \exp(\delta t) P_\beta(0, t), \\
 Q_\beta(z, t) &\rightarrow 0 \quad \text{uniformly in } t \text{ as } z \rightarrow \infty.
 \end{aligned} \tag{3.30}$$



An upper bound for  $Q_\beta(z, t)$  can be obtained in terms of the solution of the initial boundary value problem

$$\begin{aligned} \frac{\partial^2 \tilde{Q}_\beta}{\partial z^2} &= \gamma^{-1} \frac{\partial \tilde{Q}_\beta}{\partial t}, \\ \tilde{Q}_\beta(z, 0) &= 0, \quad \text{for all } z \geq 0, \\ \tilde{Q}_\beta(0, t) &= \exp(\delta t) P_\beta(0, t), \\ \tilde{Q}_\beta(z, t) &\rightarrow 0 \quad \text{uniformly in } t \text{ as } z \rightarrow \infty. \end{aligned} \tag{3.31}$$

We have (see [11])

$$Q_\beta \leq \tilde{Q}_\beta. \tag{3.32}$$

So, we obtain

$$H_\beta(z, t) \leq \exp(-2\delta t) \tilde{Q}_\beta(z, t)^2. \tag{3.33}$$

It is known (see [15]) that

$$\tilde{Q}_\beta(z, t) = \left(\frac{1}{4\pi\gamma}\right)^{1/2} z \int_0^t \frac{\exp(\delta\tau) P_\beta(0, \tau)}{(t-\tau)^{3/2}} \exp\left(-\frac{z^2}{4\gamma(t-\tau)}\right) d\tau. \tag{3.34}$$

Hence, we obtain the theorem whenever  $g$  is smooth enough. In the general case we may repeat the argument used at the end of Theorem 3.  $\square$

**Theorem 5.** *Let  $u$  be the mild solution of the initial boundary value problem (3.1)–(3.5). Then, it satisfies the spatial estimate*

$$\int_0^t \int_0^L u^2 dl d\tau \leq 8 \left(\frac{t}{\pi}\right)^2 \exp(\beta t) K_{g,\beta}(t) N(z, t)^2, \tag{3.35}$$

for all

$$z^2 > 2\delta^{1/2} t\gamma, \tag{3.36}$$

where

$$\begin{aligned} N(z, t) &= \frac{1}{2\sqrt{\pi}} \exp\left(\frac{\omega^2}{2}\right)^{1/2} \frac{\exp\left\{-\left(\frac{z}{(4\gamma t)^{1/2}} + (\delta\gamma t)^{1/2} \frac{1}{z}\right)^2\right\}}{\frac{z}{(4\gamma t)^{1/2}} + (\delta\gamma t)^{1/2} \frac{1}{z}} \\ &+ \frac{1}{2\sqrt{\pi}} \exp\left(\frac{\omega^2}{2}\right)^{1/2} \frac{\exp\left\{-\left(\frac{z}{(4\gamma t)^{1/2}} - (\delta\gamma t)^{1/2} \frac{1}{z}\right)^2\right\}}{\frac{z}{(4\gamma t)^{1/2}} - (\delta\gamma t)^{1/2} \frac{1}{z}}. \end{aligned}$$

**Proof.** Inequality (3.33) implies

$$H_\beta(z, t) \leq H_\beta(0, t)M(z, t)^2,$$

where

$$M(z, t) = \left(\frac{1}{4\pi\gamma}\right)^{1/2} z \int_0^t \tau^{-3/2} \exp\left(-\frac{z^2}{4\gamma\tau} - \mu\tau\right) d\tau.$$

Using the change of variable  $\xi^2 = z^2/(4\gamma\tau)$ , it is easily shown that

$$M(z, t) = \left(\frac{4}{\pi}\right)^{1/2} \int_{z/(4\gamma t)^{1/2}}^\infty \exp\left(-\xi^2 - \frac{\mu\xi^{-2}z^2}{4\gamma}\right) d\xi.$$

It is worth remarking from (3.15) that

$$\int_0^t \int_0^L \exp(2\lambda\tau) u^2 dl d\tau \leq 8\left(\frac{t}{\pi}\right)^2 \exp(\beta t) H_\beta(0, t) M(z, t)^2.$$

Thus, we conclude

$$\int_0^t \int_0^L u^2 dl d\tau \leq 8\left(\frac{t}{\pi}\right)^2 \exp(\beta t) H_\beta(0, t) M(z, t)^2. \tag{3.37}$$

To prove our theorem we employ a representation for the function  $M(z, t)$  given in [1]; it can be proved that

$$M(z, t) = \frac{1}{2} \left( \exp\left(\frac{\omega^2}{2}\right)^{1/2} \operatorname{erfc}\left(\frac{z}{(4\gamma t)^{1/2}} + \frac{(\delta\gamma t)^{1/2}}{z}\right) + \exp\left(-\frac{\omega^2}{2}\right)^{1/2} \operatorname{erfc}\left(\frac{z}{(4\gamma t)^{1/2}} - \frac{(\delta\gamma t)^{1/2}}{z}\right) \right),$$

where

$$\operatorname{erfc}(z) = \int_z^\infty \exp(-\xi^2) d\xi.$$

If we recall the estimate [1, p. 298]

$$\sqrt{\pi} \operatorname{erfc}(z) < \frac{1}{z} \exp(-z^2),$$

we obtain the estimate (3.35).  $\square$

It is worth remarking that the estimate (3.37) is also valid without the restriction (3.36).

#### 4. Continuous dependence with respect to the density

In this section we obtain a continuous dependence result for the solutions respect to the mass density. We denote by  $\bar{u}$  the solution of the problem determined by equation

$$\mu \Delta u + \gamma \Delta u_t = \rho u_{tt} \quad (\rho > 0) \tag{4.1}$$

and conditions (3.2)–(3.5). We could study the case in which  $\bar{u}$  differs from  $u$  on the end  $x_1 = 0$ , but since the problem is linear, we treat separately the perturbations of the mass density and the perturbations on the boundary conditions. Since the influence on  $g$  has been investigated in the previous section, we restrict our attention to the case in which  $u$  and  $\bar{u}$  satisfy the same boundary conditions. Some results concerning continuous dependence were obtained recently for the heat equation [10].

In the proof of our continuous dependence result, we will need the estimate obtained in Theorem 3 applied to the function  $\bar{u}_{tt}$ . Thus, in this section, we need assume not only that  $g$  satisfies (Hg), but that  $g_{tt}$  satisfies this condition.

**Theorem 6.** *Let  $u$  and  $\bar{u}$  be the mild solutions of the initial boundary value problem (3.1)–(3.5) but with mass density  $\rho$  in the second case. Let us also assume that  $g_{tt}$  also satisfies (Hg). Then, for arbitrary  $z \geq 0$  and  $t \geq 0$ , the difference  $u - \bar{u}$  satisfies the estimates*

$$\int_0^t \int_0^L (u - \bar{u})^2 dl d\tau \leq \frac{32t^4(\rho - 1)^2}{\pi^4 \gamma \omega(\rho)} \Omega(t) \exp(2\beta t) z \xi^{-1}(z) \exp(-\omega(\rho)z)$$

and

$$\begin{aligned} & \int_0^t \int_0^L \int_z^\infty (u - \bar{u})^2 ds d\tau \\ & \leq \frac{32t^4(\rho - 1)^2}{\pi^4 \gamma \omega(\rho)} \exp(2\beta t) \Omega(t) \int_z^\infty \chi \xi^{-1}(\chi) \exp(-\omega(\rho)\chi) d\chi, \end{aligned}$$

where

$$\begin{aligned} \omega(\rho) &= \sqrt{2 \frac{\pi^2}{L^2} + \left( \frac{\beta\rho}{\gamma} - \frac{4\rho\mu}{\gamma^2} \right)}, \\ \Omega(t) &= \frac{1}{2} \int_0^t \int_0^L \exp(-\beta\tau) \left( \frac{d}{d\tau} (\exp(\lambda\tau) g_{\tau\tau}(x_2, \tau)) \right)^2 dl d\tau, \end{aligned}$$

$$\begin{aligned} \xi^{-1}(z) &= \frac{\gamma^{-1}}{\gamma^{-1}\xi(0) - \omega(\rho)} \\ &\quad + \left( \frac{1}{\xi(0)} - \frac{\gamma^{-1}}{\gamma^{-1}\xi(0) - \omega(\rho)} \right) \exp((\omega(\rho) - \gamma^{-1}\xi(0))z), \\ \xi(0) &= 2\gamma \left( \frac{\pi}{L} + \left( \frac{\beta}{2\gamma} - 2\rho\gamma^{-2} \right) \right), \end{aligned}$$

and  $\beta$  an arbitrary constant which is greater than  $4\lambda$ .

**Proof.** As in the proof of Theorems 3 and 4, we first assume that the function  $g$  is smooth enough. If we set

$$\phi = u - \bar{u}, \tag{4.2}$$

the function  $\phi$  satisfies the equation

$$\rho\Delta\phi + \gamma\Delta\phi_t = \phi_{tt} + (\rho - 1)\bar{u}_{,tt}, \tag{4.3}$$

the initial conditions (3.5), the boundary conditions (3.2) and

$$\phi(0, x_2, t) = 0, \quad 0 \leq x_2 \leq h, \quad t > 0. \tag{4.4}$$

Through the change of variable

$$y(x_1, x_2, t) = \exp(\lambda t)\phi(x_1, x_2, t), \tag{4.5}$$

Eq. (4.3) becomes

$$\gamma\Delta y_t = y_{tt} - 2\lambda y_t + \lambda^2 y + (\rho - 1)\bar{u}_{,tt} \exp(-\lambda t). \tag{4.6}$$

Now, we define

$$Y(z, t) = \frac{1}{2} \int_0^t \int_0^L \exp(-\beta\tau)(y_\tau)^2 dl d\tau. \tag{4.7}$$

We obtain

$$\begin{aligned} \frac{\partial Y}{\partial z} &= \int_0^t \int_0^L \exp(-\beta\tau) y_\tau y_{\tau 1} dl d\tau \\ &= - \int_0^t \int_0^L \int_z^\infty \exp(-\beta\tau) \left( y_{\tau i} y_{\tau i} + \left( \frac{\beta}{2\gamma} - 2\mu\gamma^{-2} \right) (y_\tau)^2 \right) ds d\tau \\ &\quad - \frac{1}{2\gamma} \int_0^t \int_0^L \int_z^\infty \exp(-\beta\tau) ((y_\tau)^2 + \mu^2 \gamma^{-2} y^2) ds \\ &\quad - (\rho - 1)\gamma^{-1} \int_0^t \int_0^L \int_z^\infty y_\tau \bar{u}_{\tau\tau} \exp(-(\beta + \lambda)\tau) ds d\tau. \end{aligned} \tag{4.8}$$

Dropping the second integral on the right of (4.8) and making use of Schwarz’s inequality we conclude that

$$\begin{aligned} \frac{\partial Y}{\partial z} &\leq - \int_0^t \int_0^L \int_z^\infty \exp(-\beta\tau) \left( y_{\tau i} y_{\tau i} + \left( \frac{\beta}{2\gamma} - 2\mu\gamma^{-2} \right) (y_\tau)^2 \right) ds d\tau \\ &\quad + |\rho - 1| \gamma^{-1} \left( \int_0^t \int_0^L \int_z^\infty \exp(-\beta\tau) (y_\tau)^2 ds d\tau \right)^{1/2} \\ &\quad \times \left( \int_0^t \int_0^L \int_z^\infty (\bar{u}_{\tau\tau})^2 \exp(-2(\lambda + \beta)\tau) ds d\tau \right)^{1/2}. \end{aligned} \tag{4.9}$$

From the arithmetic–geometric mean inequality we obtain

$$\begin{aligned} \frac{\partial Y}{\partial z} &\leq - \int_0^t \int_0^L \int_z^\infty \exp(-\beta\tau) \left( y_{\tau i} y_{\tau i} + \left( \frac{\beta}{2\gamma} - 2\mu\gamma^{-2} \right) (y_\tau)^2 \right) ds d\tau \\ &\quad + \frac{1}{2} \gamma^{-1} \xi(z) \int_0^t \int_0^L \int_z^\infty \exp(-\beta\tau) (y_\tau)^2 ds d\tau \\ &\quad + \frac{1}{2} |\rho - 1|^2 \gamma^{-1} \xi^{-1}(z) \int_0^t \int_0^L \int_z^\infty (\bar{u}_{\tau\tau})^2 \exp(-2(\lambda + \beta)\tau) ds d\tau, \end{aligned} \tag{4.10}$$

for some positive function  $\xi(z)$ . A use of Hölder’s and Poincaré’s inequalities shows that

$$\begin{aligned} \int_0^L (y_\tau)^2 dl &= -2 \int_0^L \int_z^\infty y_\tau y_{\tau 1} ds \\ &\leq 2 \left( \int_0^L \int_z^\infty (y_\tau)^2 ds \right)^{1/2} \left( \int_0^L \int_z^\infty (y_{\tau 1})^2 ds \right)^{1/2} \\ &\leq \frac{L}{\pi} \int_0^L \int_z^\infty y_{\tau i} y_{\tau i} ds. \end{aligned}$$

Hence, we obtain the differential inequality

$$\begin{aligned} \frac{\partial Y}{\partial z} + \left( 2\left(\frac{\pi}{L} + \left(\frac{\beta}{2\gamma} - 2\mu\gamma^{-2}\right) - \gamma^{-1}\xi(z)\right) \right) Y \\ \leq \frac{(\rho - 1)^2}{2\gamma\xi(z)} \int_0^t \int_0^L \int_z^\infty (\bar{u}_{\tau\tau})^2 \exp(-(2\lambda + \beta)\tau) ds d\tau. \end{aligned} \tag{4.11}$$

The arguments used to obtain Theorem 3 can be applied to  $\bar{u}_{\tau\tau}$ . Using the inequality of Poincaré we see that

$$\begin{aligned} \int_0^t \int_0^L (\bar{u}_{\tau\tau})^2 \exp(-(2\lambda + \beta)\tau) dl d\tau \leq \int_0^t \int_0^L \exp(-\beta\tau) (\bar{u}_{\tau\tau})^2 dl d\tau \\ \leq \frac{8t^2}{\pi^2} \Omega(t) \exp(\beta t) \exp(-\omega(\rho)z). \end{aligned} \tag{4.12}$$

A quadrature shows that

$$\begin{aligned} \int_0^t \int_0^L \int_z^\infty (\bar{u}_{\tau\tau})^2 \exp(-(2\lambda + \beta)\tau) ds d\tau \\ \leq \frac{8t^2}{\omega(\rho)\pi^2} \Omega(t) \exp(\beta t) \exp(-\omega(\rho)z). \end{aligned} \tag{4.13}$$

Inserting (4.13) into (4.11) we see that

$$\begin{aligned} \frac{\partial Y}{\partial z} + \left( 2\left(\frac{\pi}{L} + \left(\frac{\beta}{2\gamma} - 2\mu\gamma^{-2}\right) - \gamma^{-1}\xi(z)\right) \right) Y \\ \leq \frac{4t^2(\rho - 1)^2}{\gamma\omega(\rho)\pi^2\xi(z)} \Omega(t) \exp(\beta t) \exp(-\omega(\rho)z). \end{aligned} \tag{4.14}$$

Let  $\xi(z)$  be the solution of Bernoulli’s equation

$$\begin{aligned} \frac{d}{dz} (\xi(z) \exp(\omega(\rho)z)) \\ = \xi(z) \exp(\omega(\rho)z) \left( 2\left(\frac{\pi}{L} + \left(\frac{\beta}{2\gamma} - 2\mu\gamma^{-2}\right) \right) - \gamma^{-1}\xi(z) \right), \end{aligned} \tag{4.15}$$

such that

$$\xi(0) = 2\gamma \left( \frac{\pi}{L} + \left( \frac{\beta}{2\gamma} - 2\mu\gamma^{-2} \right) \right). \tag{4.16}$$

Inequality (4.14) may be rewritten as

$$\frac{d}{dz} (Y(z, t)\xi(z) \exp(\omega(\rho)z)) \leq \frac{4(\rho - 1)^2 t^2 \Omega(t) \exp(\beta t)}{\gamma\omega(\rho)\pi^2}. \tag{4.17}$$

An integration gives

$$Y(z, t) \leq \frac{4(\rho - 1)^2 t^2 z \Omega(t) \exp(\beta t)}{\gamma \omega(\rho) \pi^2} \xi^{-1}(z) \exp(-\omega(\rho)z). \quad (4.18)$$

Changing variable and using Poincaré's inequality we obtain

$$\int_0^t \int_0^L \phi^2 dl d\tau \leq \frac{32(\rho - 1)^2 t^4}{\gamma \omega(\rho) \pi^4} \Omega(t) \exp(2\beta t) z \xi^{-1}(z) \exp(-\omega(\rho)z). \quad (4.19)$$

The second estimate follows from a direct quadrature.

As at the end of Theorems 3 and 4, we may approximate the solution by a sequence of solutions satisfying the regularity conditions assumed at the beginning of the proof. We reach the final result passing to the limit and using the fact that the estimate does not depend on the regularity condition.  $\square$

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