

On the von Neumann problem and the approximate controllability of Stackelberg-Nash strategies for some environmental problems

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Dedicated to the memory of Jacques-Louis Lions

Abstract. Two problems arising in Environment are considered. The first one concerns a conjecture posed by von Neumann in 1955 on the possible modification of the albedo in order to control the Earth surface temperature. The second one is related to the approximate controllability of Stackelberg-Nash strategies for some optimization problems as, for instance, the pollution control in a lake. The results of the second part were obtained in collaboration with Jacques-Louis Lions.

Sobre el problema de von Neumann y la controlabilidad aproximada de estrategias de Stackelberg-Nash para ciertos problemas de medio ambiente

Resumen. Se consideran dos problemas planteados en medio ambiente. El primero de ellos se refiere a una conjetura propuesta por von Neumann en 1955 sobre la posibilidad de actuar sobre el albedo para modificar la temperatura superficial de la Tierra. El segundo problema aborda la controlabilidad aproximada de estrategias de Stackelberg-Nash para la optimización en temas tales como el control de la polución en un lago. Los resultados de la segunda parte fueron obtenidos en colaboración con Jacques-Louis Lions.

1. Introduction

In this paper we consider several environmental problems under the common formulation of a general control problem of the type

$$\frac{\partial \mathbf{y}}{\partial t} + \mathcal{A}(\mathbf{y}) = \text{sources} + \text{sinks} + \text{actions} \quad (1)$$

and with the *initial state* supposed to be given

$$\mathbf{y}(x, 0) = \mathbf{y}_0(x). \quad (2)$$

Here by *actions* we mean some *active controls* and they are the unknowns of the problem. Suppose, for instance, that we are not satisfied with the initial state and that it would be better to be near a given (*ideal*

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target state) y^T . Given $T > 0$, can we “drive the system” (by choosing the actions) in such a manner that $y(\cdot, T)$ let as close as possible to y^T ? When our interest is fixed in a concrete target state, y^T , the problem can be understood as a typical *inverse problem*. Nevertheless, if our interest is in to know the answer for a large family of target states y^T belonging to some functional space X then we arrive to the notion of *approximate controllability* in X .

Before to be more precise, let us mention that the *AMS Mathematical Classification* devotes a special issue to related problems arising in *Environmental economics* (see 91B76). Many different examples of antropogenerated actions are being applying today. Most of them are *local actions* as for, instance, the “cloud seeding” (see, e.g., Dennis [6]). In a more global level, the present global actions (such as limitations laws for the atmospheric pollution of Rio, Kyoto, the proposals by the IPGC, etc.) coexist with some speculations (see, e.g. the article by Thomson [37]) of unclear interest breaking the *Precautionary Principle* (see Harremoës [20]).

The first part of this paper deals with an inverse problem suggested by John von Neumann in 1955 ([30]):

Microscopic layers of colored matter spread on an icy surface, or in the atmosphere above one, could inhibit the reflection-radiation process, melt the ice and change the local climate. Probably intervention in atmospheric and climate matters will come in a few decades, and will unfold on a scale difficult to imagine at present.

The idea of von Neumann was to act on the atmospheric climate by acting on the albedo. This type of problems corresponds to the case of a *single control* v , in the terminology of Control and Games Theories.

We present some recent results on a mathematical formulation expressed in terms of the, so called, *Energy Balance Models*, introduced, in 1969, independently by M. I. Budyko and W. D. Sellers. Roughly speaking, the problem is to find $v(x)$ such that $y(T : v) = y^T$ with $y(T : v)$ solution of the problem

$$(P_v) \begin{cases} y_t - \Delta y + By + C = QS(x)(\beta(y) + v(x)H(y)) & \text{in } (\mathcal{M} - \mathcal{I}) \times (0, T), \\ y = u_i & \text{on } \Gamma \times (0, T), \\ y(0, \cdot) = y_0(\cdot) & \text{on } \mathcal{M} - \mathcal{I}. \end{cases}$$

The meaning of the different terms involved in the above formulation will be given in Section 2.

In the second part of the paper we consider a different environmental control problem involving several players (non necessarily cooperating among them) according to a formulation introduced, in Economy, by H. von Stackelberg in 1934 ([36]). As a typical example (an academic scenario) we can mention the problem of to maintain clean a resort lake represented by an open and bounded set Ω of \mathbb{R}^3 . The state of the system is denoted by \mathbf{y} . It is a vector function $\mathbf{y} = (y_1, \dots, y_N)$, each y_i being a function of time, t , and space, $x \in \Omega$. The y_i 's correspond to concentrations of various chemical products or of living organisms in the lake Ω . The y_i 's are therefore given by the solution of a set of *diffusion-convection equations*. We assume the presence of several local agents or local plants P_1, P_2, \dots, P_N . Each plant decide its policy (here represented by an unknown function $w_i(t, x)$). We also assume the existence of a different action (the unknown $v(t, x)$) taken by a representative authority, or general manager (*leader*), in contrast to the rest of the players (*followers*).

The general goal of the manager is “to drive the state of the system” at time T , $\mathbf{y}(T : v)$, as close as possible to an ideal state \mathbf{y}^T , by means of the control v . Each plant has (essentially) the same goal but it will be particularly careful to the state \mathbf{y} near its location. To express that we introduce a smooth function ρ_i given in $\overline{\Omega}$ such that

$$\rho_i(x) \geq 0, \rho_i = 1 \text{ near the location of } P_i. \quad (3)$$

Then P_i will try to choose w_i such that the state at time T , $\mathbf{y}(x, T)$, will be “close” to $\rho_i \mathbf{y}^T$ and to achieve this at *minimum cost*. This leads to the introduction of

$$J_i(v; w_1, \dots, w_N) = \frac{1}{2} \|w_i\|^2 + \frac{\alpha_i}{2} \|\rho_i(\mathbf{y}(\cdot, T) - \mathbf{y}^T)\|^2, \quad (4)$$

where $\|w_i\|$ represents the cost of w_i , α_i is a given positive constant and $\|\rho_i(\mathbf{y}(\cdot, T) - \mathbf{y}^T)\|$ is a measure of the “localized distance” between the actual state at time T and the desired state \mathbf{y}^T .

The “local” controls w_1, \dots, w_N assume that *the leader* has made a choice v and they try to find a *Nash equilibrium* ([29]) of their cost J_i , *i.e.*, they look for w_1, \dots, w_N (as functions of v) such that

$$\left. \begin{aligned} J_i(v; w_1, \dots, w_{i-1}, w_i, w_{i+1}, \dots, w_N) &\leq J_i(v; w_1, \dots, w_{i-1}, \widehat{w}_i, w_{i+1}, \dots, w_N), \\ \forall \widehat{w}_i, \text{ for } i &= 1, \dots, N. \end{aligned} \right\} \quad (5)$$

If $\mathbf{w}=(w_1, \dots, w_N)$ satisfies (5) one says it is a *Nash equilibrium*.

We assume that the leader v wants now that the *global* state (*i.e.* the state $\mathbf{y}(\cdot, T)$ in the whole domain Ω) to be *as close as possible* from \mathbf{y}^T . This will be possible, for any given function \mathbf{y}^T , if the problem is *approximately controllable*, *i.e.* if

$$\left. \begin{aligned} \mathbf{y}(x, t; v; w_1, \dots, w_N) &\text{ describes a dense subset of the given} \\ \text{state space when } v &\text{ spans the set of all controls available to the leader.} \end{aligned} \right\} \quad (6)$$

We give some sufficient conditions for the existence (and uniqueness) of a Nash equilibrium. We shall prove also that, if there is existence and uniqueness of a Nash equilibrium for the followers, then the leader can control the system (in the sense of approximate controllability). Most of the results of Section 3 are originated from the paper [11] written in collaboration with Jacques-Louis Lions. We also present here a new proof, of a constructive nature, of the approximate controllability. Some remarks on the case of nonlinear state equations are also given.

2. The von Neumann problem

We shall formulate the von Neumann problem in terms of the so called Energy Balance Models, introduced, independently by M. I. Budyko [4] and W. D. Sellers [35] (some pioneering model is due to S. Arrhenius in 1896). Such type of climatological models have a diagnostic character and intended to understand the evolution of the global climate on along time scale. Their main characteristic is the high sensitivity to the variation of solar and terrestrial parameters. They have been used in the study of the *Milankovitch theory* of the ice-ages (see, e.g. Mengel, Short and North [27]). They study a distribution of temperature, $y(x, t)$, which is expressed pointwise after some averaging process in space (the spatial variable x is in a small neighborhood $B(x)$ in the Earth’s surface) and in time (on a small interval $(t - \tau, t + \tau)$)

$$y(x, t) = \frac{1}{2\tau |B(x)|} \int_{t-\tau}^{t+\tau} \int_{B(x)} T(a, s) da ds.$$

The pointwise temperature $T(a, s)$ is obtained from the thermodynamics equation of the *atmosphere primitive equations*(see *e.g.*, Lions, Temam and Wang [26] for a mathematical study of those equations and Kiehl [22] for the application of averaging processes in this context). More simply, the energy balance model can be formulated by using the energy balance on the Earth’s surface: internal energy flux variation= $R_a - R_e + D$, where R_a (respectively R_e) represents the absorbed solar (resp. the emitted terrestrial energy flux) and where D is the horizontal heat diffusion. By identifying the Earth’s surface with a compact Riemannian manifold without boundary \mathcal{M} (for instance, the two-sphere S^2), the distribution of temperature, $y(x, t)$, becomes a function of the spatial x and t time variables. The time scale is considered relatively long. The *absorbed energy* R_a depends on the planetary *coalbedo* β . The coalbedo function represents the fraction of the incoming radiation flux which is absorbed by the surface. In ice-covered zones, reflection is greater than over oceans, therefore, the coalbedo is smaller. One observes that there is a sharp transition between zones of high and low coalbedo. In the energy balance climate models, a main change of the coalbedo occurs in a neighborhood of a critical temperature for which ice become white, usually taken as $y = -10^0C$. The coalbedo can be modelled by different monotone increasing fuctions (discontinuous in case of *Budyko models* and Lipschitz continuous for *Sellers model*). A more realistic albedo parametrization can be obtained

by assuming that the coalbedo fuction $\beta(x, y)$ also depends on the spatial coordinates of each point of the Earth (specially on its latitude: see [19], Section 3.3)

$$\beta(u) = \beta(x, u) = \begin{cases} \beta_i(x) & u < u_i, \\ \beta_i(x) + \left(\frac{u - u_i}{u_w - u_i}\right)(\beta_w(x) - \beta_i(x)) & u_i \leq u \leq u_w, \\ \beta_w(x) & u > u_w, \end{cases} \quad (7)$$

with $0 < u_i < u_w < 1$ (the coalbedo values for the ice-covered zone and the free-ice zone which can be estimated by observation from satellites).

To simplify the presentation we assume that the internal energy flux variation is simply given as the product of the heat capacity c (a given constant which can be assumed equal to one by rescaling the variables) and the partial derivative of the temperature y with respect to the time. In both models, the absorbed energy is given by $R_a = QS(x)\beta(x, y)$ where $S(x)$ is the *insolation function* and Q is the so-called *solar constant*.

The Earth's surface and atmosphere, warmed by the Sun, reemit part of the absorbed solar flux as an infrared long-wave radiation. This energy R_e is represented, in the Budyko model, according to the Newton cooling law, that is,

$$R_e = By + C. \quad (8)$$

Here, B and C are positive parameters, which are obtained by observation, and can depend on the *greenhouse effect* (however, in the Sellers model, R_e is expressed according to the Stefan–Boltzman law $R_e = \sigma y^4$, where σ is called *emissivity constant* and now y is in Kelvin degrees).

The *heat diffusion* D is given by the divergence of the conduction heat flux F_c and the advection heat flux F_a . Fourier's law expresses $F_c = k_c \nabla y$ where k_c is the *conduction coefficient*. The advection heat flux is given by $F_a = \mathbf{v} \cdot \nabla y$ and it is known (see, e.g., Ghil and Childress [17]) that, to the level of the planetary scale, it can be modeled in terms of $k_a \nabla y$ for a suitable diffusion coefficient k_a . So, $D = \text{div}(k \nabla y)$ with $k = k_c + k_a$. In the pioneering models, the diffusion coefficient k was considered as a positive constant.

A mathematical study of actions on the emmissions can be found in Díaz ([9]). Concerning the von Neumann conjecture, a possible formulation of the action proposed on the coalbedo could be the following

$$\beta(x, u; v) = \begin{cases} \beta_i & u < u_i, \\ \beta_i + \left(\frac{u - u_i}{u_w - u_i}\right)(\beta_w + v(x)\chi_\omega(x) - \beta_i) & u_i \leq u \leq u_w, \\ \beta_w + v(x)\chi_\omega(x) & u > u_w, \end{cases} \quad (9)$$

with $\beta_w > \beta_i$ constants and with $\chi_\omega(x)$ being the characteristic function of a small region ω where the albedo modification is taking place. Notice that we can write $\beta(x, u; v) = \beta(u) + v(x)\chi_\omega(x)H(u)$ with

$$H(u) = \begin{cases} 0 & u < u_i, \\ \left(\frac{u - u_i}{u_w - u_i}\right) & u_i \leq u \leq u_w, \\ 1 & u > u_w. \end{cases} \quad (10)$$

Some additional conditions are needed in order to simplify the formulation of the question. The first one is that we shall assume that the local modifications on the albedo does not introduce any important change on the region occupied by the polar and perpetuum ices (i.e. on the set $\mathcal{I} = \{x \in \mathcal{M} : y(x, 0) \geq u_i\}$). Our arguments will be valid also for the opposite case in which the modifications are made unically on a neighborhood of the region $\{x \in \mathcal{M} : y(x, 0) \leq u_w\}$. So, in the rest of this section we shall assume that $\omega = \mathcal{M} - \mathcal{I}$.

Given a target temperature function $y^T(x)$ (for instance, the temperature distribution before the industrial era), we consider the problem of to find $v(x)$ such that $y(T : v) = y^T$, with $y(T : v)$ solution of

$$(\mathcal{P}_v) \begin{cases} y_t - \Delta y + By + C = QS(x)(\beta(y) + v(x)H(y)) & \text{in } (\mathcal{M} - \mathcal{I}) \times (0, T), \\ y = u_i & \text{on } \Gamma \times (0, T), \\ y(0, \cdot) = y_0(\cdot) & \text{on } \mathcal{M} - \mathcal{I}, \end{cases}$$

where $\Gamma = \partial\mathcal{I}$. Our main goal is to give a positive answer by means of a suitable application of a fixed point argument.

We point out that the controls v (i.e. the albedo modification) must satisfy the constraint $\beta(y) + v(x)H(y) \in [0, 1]$. If, in particular, we want to reach a target temperature function $y^T(x)$ considerably less than the present (i.e. without any voluntary action: $v = 0$) then the relevant controls must take negative values (otherwise there is a growth of the temperature). So we shall assume that

$$v(x) \in [-\mu, 0], \quad x \in \mathcal{M} - \mathcal{I}, \quad \text{with } \mu = \beta_w - \beta_i. \quad (11)$$

Notice that by the *strong maximum principle* we can assume that $y(x, t : v) > u_i$ for any control v and almost every point $(x, t) \in (\mathcal{M} - \mathcal{I}) \times (0, T)$.

It is important to see that the target state $y^T(x)$ must be in good correspondence with the limitations on the controls. So, by the maximum principle, given v verifying (11), necessarily

$$y(T : -\mu) \leq y(T : v) \leq y(T : 0) \text{ in } \mathcal{M} - \mathcal{I},$$

and so, if $y(T : v) = y^T$, we get the necessary condition on y^T :

$$y(T : -\mu) \leq y^T \leq y(T : 0) \text{ in } (\mathcal{M} - \mathcal{I}).$$

Our goal is to made explicit some class of functions y^T in the attainability set associated to such controls. Notice that for fixed $\gamma, \delta > 0$, if we assume $y^T = u_i$ on Γ , the existence of the searched control $v(x)$ is reduced to the existence of a fixed point for the map $v \rightarrow \mathcal{T}v$ given by

$$(\mathcal{T}v)(x) = \frac{y_t(T : v) + By(T : v) + C + QS(x)(v(\gamma y(T : v) + \delta) - \beta(y(T : v))) - \Delta y^T}{QS(x)(H(y(T : v)) + \gamma y(T : v) + \delta)}, \quad x \in \mathcal{M} - \mathcal{I}.$$

Indeed, if v is such fixed point then we get that $\Delta y^T = \Delta y(T : v)$ and functions y^T and $y(T : v)$ satisfy the same boundary conditions. So we deduce that $y^T = y(T : v)$ in $\mathcal{M} - \mathcal{I}$.

Notice that parameters γ and $\delta > 0$ have been introduced in order to avoid the indetermination arising when the denominator of $\mathcal{T}v$ vanishes (a more technical reason will be mentioned later). We shall take $\gamma > 0$ large enough insuring that function $v \rightarrow v(\gamma y(T : v) + \delta) - \beta(y(T : v))$ is non-decreasing when v verifies (11). By the maximum principle, $y(T : v)$ depends monotonically from v and so it will be enough to assume that

$$\gamma > \frac{\beta_w - \beta_i}{u_w - u_i}. \quad (12)$$

To keep positive the denominator of \mathcal{T} we shall assume that

$$\delta > \gamma(-u_i). \quad (13)$$

A result giving a positive answer to the von Neumann conjecture is the following:

Theorem 1 Assume (12), (13) and let $y_0 \in C^{2,\alpha}(\overline{\mathcal{M} - \mathcal{I}})$ verifying the compatibility condition $y_0 = u_i$ on Γ and such that

$$\Delta y_0 - By_0 - C + QS(x)\beta(y_0) \leq 0 \text{ on } \mathcal{M} - \mathcal{I}. \quad (14)$$

Let $y^T \in C^{2,\alpha}(\overline{\mathcal{M} - \mathcal{I}})$ such that $y^T = u_i$ on Γ and satisfying that

$$\Delta y(T : 0) \leq \Delta y^T \text{ on } \mathcal{M} - \mathcal{I}, \quad (15)$$

$$\Delta y^T \leq \Delta y(T : 0) + QS(x)(\beta_w - \beta_i)(H(y(T : 0)) + \delta) \text{ on } \mathcal{M} - \mathcal{I}. \quad (16)$$

Then, there exists $v \in C^{0,\alpha}(\overline{\mathcal{M} - \mathcal{I}})$ with $v(x) \in [-\mu, 0]$, $\forall x \in (\mathcal{M} - \mathcal{I})$ such that $y(T : v) = y^T$ in $\mathcal{M} - \mathcal{I}$.

PROOF. Let us prove that operator \mathcal{T} is monotone but reversing the order. Let $v_1 \leq v_2$. Then

$$y_{1t} - \Delta y_{1t} + B y_{1t} + C = QS(x)(\beta(y_1) + v_1(x)(H(y_1) + \delta)) \leq QS(x)(\beta(y_1) + v_2(x)(H(y_1) + \delta))$$

and so, by the comparison principle for the problem satisfied by y_2 , we deduce that $y_1 \leq y_2$ on $(\mathcal{M} - \mathcal{I}) \times (0, T)$. Let us see that, in fact, we also have that $y_{1t} \leq y_{2t}$. To do that, we approximate β and H by smooth functions (which we denote again by β and H). By differentiating in the equation we get that

$$y_{1tt} - \Delta y_{1t} + B y_{1t} = QS(x)[\beta'(y_1) + v_1(x)H'(y_1)]y_{1t}.$$

By the maximum principle and assumption (14) we get that $y_{1t}(t, v) \leq 0$ for any v and almost every $t \in (0, T)$. In consequence, since $\beta'(r) + v(x)H'(r)$ is decreasing when $v(x) \in [-\mu, 0]$ and $r \geq u_i$, we deduce that

$$\begin{aligned} y_{1tt} - \Delta y_{1t} + B y_{1t} &= QS(x)[\beta'(y_1) + v_1(x)H'(y_1)]y_{1t} \\ &\leq QS(x)[\beta'(y_2) + v_1(x)H'(y_2)]y_{1t}. \end{aligned}$$

On the other hand, since $y_{1t} = y_{2t} = 0$ on $\Gamma \times (0, T)$ and $y_{1t}(0, x) \leq y_{2t}(0, x)$ for $x \in \mathcal{M} - \mathcal{I}$, from the comparison principle, we conclude that $y_{1t} \leq y_{2t}$ in $(\mathcal{M} - \mathcal{I}) \times (0, T)$. Notice that this inequality remains true when passing to the limit in the process of approximation of β and H . Now, to check that \mathcal{T} is monotone it is enough to observe that from (12) and (15) we deduce that

$$\begin{aligned} &y_t(T : v) + B y(T : v) + C + QS(x)(v(\gamma y(T : v) + \delta) - \beta(y(T : v))) \\ &\leq y_t(T : 0) + B y(T : 0) + C - QS(x)\beta(y(T : 0)) = \Delta y(T : 0) \leq \Delta y^T, \end{aligned}$$

and so, the numerator of \mathcal{T} is less or equal to zero and the conclusion is obtained from (12) and the monotonicity of $H(y(T : v))$ with respect to v .

Now, condition (15) implies that $\mathcal{T}(-\mu) \leq 0$ and assumption (16) leads to $-\mu \leq \mathcal{T}(0)$. Then, the “interval of functions” $[-\mu, 0] = \{w \in C^{0,\alpha}(\mathcal{M} - \mathcal{I}) \text{ with } -\mu \leq w(x) \leq 0, \forall x \in \mathcal{M} - \mathcal{I}\}$ is invariant by \mathcal{T} , i.e. $\mathcal{T}([-\mu, 0]) \subset [-\mu, 0]$.

Finally, the solution regularity $L^\infty(0, T : W^{2,p}(\mathcal{M} - \mathcal{I})) \cap H^1(0, T : L^\infty(\mathcal{M} - \mathcal{I}))$, for any $p > 1$, implies that \mathcal{T} is relatively compact and the conclusion is obtained by the Amann fixed point theorem [1]. ■

Remark 1 For some related results for other inverse problems see Choulli [5] and Zeghal [38] (see many other previous works in their list of references). We also point out that the successive iterations of the operator \mathcal{T} applied to $v = 0$ and $v = -\mu$, respectively, lead to a constructive algorithm of the fixed points of \mathcal{T} (non necessarily be unique). ■

Remark 2 Notice that, by taking the inverse operator to the Laplacian (with the boundary condition given in (P_v)) in both terms of the condition (15) we get that necessarily $y(T : 0) - \theta \leq y^T \leq y(T : 0)$ in $(\mathcal{M} - \mathcal{I})$, with $\theta(x) := (\beta_w - \beta_i)(-\Delta)^{-1}[QS(x)(H(y(T : 0)) + \delta)]$. ■

3. The Stackelber-Nash strategies for the approximate controllability in some parabolic problems

Let A be a second order elliptic operator in Ω :

$$A\varphi = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{i,j}(x) \frac{\partial \varphi}{\partial x_j} \right) + \sum_{i=1}^N a_i(x) \frac{\partial \varphi}{\partial x_i} + a_0(x)\varphi, \tag{17}$$

where all coefficients are smooth enough and where

$$\sum_{i,j=1}^N a_{i,j}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^N \xi_i^2, \quad \alpha > 0, \quad x \in \overline{\Omega}. \quad (18)$$

We assume that the state equation is given by

$$\frac{\partial y}{\partial t} + Ay = v\chi + \sum_{i=1}^N w_i\chi_i \quad (19)$$

where

$$\left. \begin{array}{l} \chi \text{ is the characteristic function of } \mathcal{O} \subset \Omega \\ \chi_i \text{ is the characteristic function of } \mathcal{O}_i \subset \Omega. \end{array} \right\} \quad (20)$$

The control function $v(x, t)$ of the leader is distributed in \mathcal{O} and the control function $w_i(x, t)$ of the follower “ i ” is distributed in \mathcal{O}_i . We assume that the *initial state* is

$$y(x, 0) = 0, \quad x \in \Omega. \quad (21)$$

Since the system is *linear* there is no restriction in assuming the initial state to be zero, in the same way as there is no restriction in assuming in (19) that there is not any source neither sinks (as a function $f(x, t)$). We assume that the *boundary conditions* are

$$y = 0 \text{ on } \partial\Omega \times (0, T). \quad (22)$$

We introduce now functions ρ_i such that

$$\left. \begin{array}{l} \rho_i \in L^\infty(\Omega), \quad \rho_i \geq 0, \\ \rho_i = 1 \text{ in a domain } \mathcal{O}_i \subset \Omega, \end{array} \right\} \quad (23)$$

and we define the cost function J_i

$$J_i(v; w_1, \dots, w_N) = \frac{1}{2} \int_0^T \int_{\mathcal{O}_i} w_i^2 dx dt + \frac{\alpha_i}{2} \|\rho_i y(T; v, \mathbf{w}) - \rho_i y^T\|^2, \quad (24)$$

where $\|\cdot\|$ is the norm in $L^2(\Omega)$. We assume that $v \in L^2(\mathcal{O} \times (0, T))$, $w_i \in L^2(\mathcal{O}_i \times (0, T))$ and that $y(x, t; v, \mathbf{w})$ is the solution of (19), (21), (22).

Given $v \in L^2(\mathcal{O} \times (0, T))$, we now define

$$\left. \begin{array}{l} \mathbf{w} = \{w_1, \dots, w_N\}, \text{ a Nash equilibrium for the cost} \\ \text{functions } J_1, \dots, J_N \text{ given by (24).} \end{array} \right\} \quad (25)$$

We will show that this *Nash equilibrium* can be defined as a function of v :

$$\mathbf{w} = \mathbf{w}(v) \text{ or } w_i = w_i(v), \quad i = 1, \dots, N. \quad (26)$$

We then replace in (19) w_i by $w_i(v)$:

$$\frac{\partial y}{\partial t} + Ay = v\chi + \sum_{i=1}^N w_i(v)\chi_i \quad (27)$$

subject to (21) and (22). System (27), (21) and (22) admits a unique solution $y(x, t; v, \mathbf{w}(v))$. We have

Theorem 2 *Assume*

$$\text{the set of inequalities (5) admits an unique solution (a Nash equilibrium).} \quad (28)$$

Then, when v spans $L^2(\mathcal{O} \times (0, T))$, the functions $y(\cdot, T; v, \mathbf{w}(v))$ describe a dense subset of $L^2(\Omega)$. In other words

$$\left. \begin{array}{l} \text{there is approximate controllability of the system} \\ \text{when a strategy of the Stackelberg-Nash type is followed.} \end{array} \right\} \quad (29)$$

3.1. A non constructive proof of Theorem 2.

We start with some considerations about the Nash equilibrium. We have (5) iff

$$\int_0^T \int_{O_i} w_i \widehat{w}_i dx dt + \alpha_i \int_{\Omega} \rho_i^2 (y(T; v, \mathbf{w}) - y^T) \widehat{y}_i(T) dx = 0, \forall \widehat{w}_i, \quad (30)$$

where \widehat{y}_i is defined by

$$\begin{cases} \frac{\partial \widehat{y}_i}{\partial t} + A \widehat{y}_i = \widehat{w}_i \chi_i, & \text{in } \Omega \times (0, T), \\ \widehat{y}_i(0) = 0, & \text{in } \Omega, \\ \widehat{y}_i = 0, & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (31)$$

In order to express (30) in a convenient form, we introduce the adjoint state p_i defined by

$$\begin{cases} -\frac{\partial p_i}{\partial t} + A^* p_i = 0 & \text{in } \Omega \times (0, T), \\ p_i(x, T) = \rho_i^2(x) (y(x, T; v, \mathbf{w}) - y^T(x)), & \text{in } \Omega, \\ p_i = 0, & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (32)$$

where A^* stands for the adjoint of A . If we multiply (32) by \widehat{y}_i and if we integrate by parts, we find

$$\int_{\Omega} \rho_i^2 (y(T; v, \mathbf{w}) - y^T) \widehat{y}_i(T) dx = \int_0^T \int_{\Omega} p_i \widehat{w}_i \chi_i dx dt,$$

so that (30) becomes

$$\int_0^T \int_{O_i} (w_i + \alpha_i p_i) \widehat{w}_i dx dt = 0, \forall \widehat{w}_i,$$

i.e.

$$w_i + \alpha_i p_i \chi_i = 0. \quad (33)$$

Then, if $\mathbf{w} = (w_1, \dots, w_N)$ is a *Nash equilibrium*, we have

$$\begin{cases} \frac{\partial y}{\partial t} + Ay + \sum_{i=1}^N \alpha_i p_i \chi_i = v \chi, & \text{in } \Omega \times (0, T), \\ -\frac{\partial p_i}{\partial t} + A^* p_i = 0, \quad i = 1, \dots, N, & \text{in } \Omega \times (0, T), \\ y(0) = 0, \quad p_i(x, T) = \rho_i^2(x) (y(x, T; v, \mathbf{w}) - y^T(x)), & \text{in } \Omega, \\ y = 0, \quad p_i = 0, & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (34)$$

We recall that here we are assuming the existence and uniqueness of a *Nash equilibrium* (see (28)).

PROOF OF THEOREM 2. We want to show that the set described by $y(\cdot, T; v)$ is dense in $L^2(\Omega)$, where y is the solution given by (34) and when v spans $L^2(\mathcal{O} \times (0, T))$. We do not restrict the problem by assuming that $y^T \equiv 0$. Let f be given in $L^2(\Omega)$ and let us assume that

$$(y(\cdot, T; v), f) = 0, \forall v \in L^2(\Omega). \quad (35)$$

We want to show that $f \equiv 0$. Let us introduce the solution $\{\varphi, \psi_1, \dots, \psi_N\}$ of the adjoint system

$$\begin{cases} -\frac{\partial \varphi}{\partial t} + A^* \varphi = 0, \\ \frac{\partial \psi_i}{\partial t} + A \psi_i = -\alpha_i \varphi \chi_i, \\ \varphi(T) = f + \sum_i \psi_i(T) \rho_i^2, \\ \psi_i(0) = 0, \\ \varphi = 0, \quad \psi_i = 0 \text{ in } \partial\Omega \times (0, T). \end{cases} \quad (36)$$

We multiply the first (resp. the second) equation in (36) by y (resp. p_i). We obtain

$$\begin{cases} -(f + \sum \psi_i(T)\rho_i^2, y(T)) + \int_0^T \int_{\Omega} \varphi(\frac{\partial y}{\partial t} + Ay) dx dt + \sum_i (\psi_i(T), p_i(T)) \\ + \sum_i \int_0^T \int_{\Omega} \psi_i(-\frac{\partial p_i}{\partial t} + A^* p_i) dx dt = - \sum_i \alpha_i \int_0^T \int_{\Omega} \varphi p_i \chi_i dx dt. \end{cases} \quad (37)$$

Using (34) (where $y^T \equiv 0$) (37) reduces to

$$-(f, y(T)) + \int_0^T \int_{\Omega} \varphi v \chi dx dt = 0. \quad (38)$$

Therefore if (35) holds, then

$$\varphi = 0 \text{ on } \mathcal{O} \times (0, T). \quad (39)$$

Using the Unique Continuation Theorem (see Mizohata [28] or Saut and Scheurer [34]) it follows from (36)₁ and (39) that

$$\varphi = 0 \text{ on } \Omega \times (0, T). \quad (40)$$

Then (36)₂, (36)₄ and $\psi_i = 0$ in $\partial\Omega \times (0, T)$ imply that

$$\psi_i = 0 \text{ in } \Omega \times (0, T), \forall i = 1, \dots, N, \quad (41)$$

so that (36)₃ gives $f \equiv 0$. ■

3.2. A criterion for the existence and uniqueness of Nash equilibria.

We consider the functionals (24). Let us define

$$\begin{cases} \mathcal{H}_i = L^2(\mathcal{O}_i \times (0, T)), \mathcal{H} = \prod_{i=1}^N \mathcal{H}_i, \\ L_i \widehat{w}_i = \widehat{y}_i(T) \text{ (cf. (31)) which defines } L_i \in L(\mathcal{H}_i; L^2(\Omega)). \end{cases} \quad (42)$$

Since v is fixed, one can write

$$y(T; v, \mathbf{w}) = \sum_{i=1}^N L_i w_i + z^T, \quad z^T \text{ fixed.} \quad (43)$$

With these notations (24) can be rewritten

$$J_i(v; \mathbf{w}) = \frac{1}{2} \|w_i\|_{\mathcal{H}_i}^2 + \frac{\alpha_i}{2} \left\| \rho_i \left(\sum_j L_j w_j - \eta^T \right) \right\|^2 \quad (44)$$

where $\eta^T = y^T - z^T$. Then $\mathbf{w} \in \mathcal{H}$ is a *Nash equilibrium* iff

$$(w_i, \widehat{w}_i)_{\mathcal{H}_i} + \alpha_i (\rho_i (\sum_j L_j w_j - \eta^T), \rho_i L_i \widehat{w}_i) = 0, \forall i = 1, \dots, N, \forall \widehat{w}_i. \quad (45)$$

or

$$w_i + \alpha_i L_i^* (\rho_i^2 \sum_{j=1}^N L_j w_j) = \alpha_i L_i^* (\rho_i^2 \eta^T), \forall i = 1, \dots, N, \quad (46)$$

(where $L_i^* \in \mathcal{L}(L^2(\Omega); \mathcal{H}_i)$ is the adjoint of L_i), or equivalently

$$\begin{cases} \mathbf{Lw} = \text{given in } \mathcal{H}, \mathbf{L} \in \mathcal{L}(\mathcal{H}; \mathcal{H}) \\ (\mathbf{Lw})_i = w_i + \alpha_i L_i^* (\rho_i^2 \sum_{j=1}^N L_j w_j). \end{cases} \quad (47)$$

Then we have

Proposition 1 Assume that

$$\alpha_i = \alpha, \forall i, \tag{48}$$

and that

$$\alpha \|\rho_i - \rho_j\|_{L^\infty(\Omega)} \|\rho_i\|_{L^\infty(\Omega)} \text{ is small enough, for any } i, j = 1, \dots, N. \tag{49}$$

Then \mathbf{L} is invertible. In particular there is a unique Nash equilibrium of (24).

Remark 3 Of course, if $N = 1$ the situation is much simpler. In that case

$$(\mathbf{L}w, w) = \|w_1\|^2 + \alpha_1 \|\rho_1 L_1 w_1\|^2.$$

Hence \mathbf{L} is *coercive* and so the existence and uniqueness of a *minimum* w of $J_1(v; w)$, when v is fixed, is a classical result. ■

PROOF OF PROPOSITION 1 In the general case $N > 1$, one has

$$(\mathbf{L}\mathbf{w}, \mathbf{w}) = \sum_i \|w_i\|_{\mathcal{H}_i}^2 + \sum \alpha_i (\rho_i \sum_j L_j w_j, \rho_i L_i w_i). \tag{50}$$

Then one can write

$$(\mathbf{L}\mathbf{w}, \mathbf{w}) = \sum_{i=1}^N \|w_i\|_{\mathcal{H}_i}^2 + \alpha \left\| \sum_{i=1}^N \rho_i L_i w_i \right\|^2 + \alpha \sum_{i,j=1}^N (\rho_i - \rho_j)^2 (L_j w_j, \rho_i L_i w_i). \tag{51}$$

Applying Young's inequality it follows that, under hypothesis (49), \mathbf{L} is coercive, i.e.

$$(\mathbf{L}\mathbf{w}, \mathbf{w}) \geq \gamma \|\mathbf{w}\|_{\mathcal{H}}^2, \text{ for some } \gamma > 0. \tag{52}$$

The conclusion is now consequence of the Lax-Milgram theorem. ■

Remark 4 Assumption (49) is certainly satisfied if $\rho_i = \rho \forall i$, in which case there is *only* one function $J_i = J_1 \forall i$, and we are back to Remark 3 (with $\mathbf{w} = (w_1, \dots, w_N)$). ■

Remark 5 It is possible to show (see [11]) that the assumptions are optimal in some suitable sense. See also [32] for some related results. ■

Remark 6 It is easy to see that, in fact, there are infinite controls v leading to the approximate controllability. ■

3.3. The optimal leader action: a constructive proof.

Given $\delta > 0$, we want to find the best leader control v in the sense of solving the problem

$$\inf_{v \in L^2(\mathcal{O} \times (0, T))} \left\{ \frac{1}{2} \int_{\mathcal{O} \times (0, T)} |v|^2 dx dt, y(T, v) \in y^T + \delta B_{L^2(\Omega)} \right\},$$

where B is the unit ball of $L^2(\Omega)$.

Theorem 3 i) *The minimum v is given by $v = \varphi\chi$ from the unique solution $\{y, p_1, \varphi, \psi_1\}$ of the “Optimality System”*

$$\begin{cases} \frac{\partial y}{\partial t} + Ay + \alpha_1 p_1 \chi_1 = v\chi, \\ -\frac{\partial p_1}{\partial t} + A^* p_1 = 0, \\ -\frac{\partial \varphi}{\partial t} + A^* \varphi = 0, \\ \frac{\partial \psi_1}{\partial t} + A\psi_1 = -\alpha_1 \varphi \chi_1, \\ y = p_1 = \varphi = \psi_1 = 0 \text{ on } \Sigma, \\ \varphi(T) = f, p_1(T) = p_1^2(y(T; v, w_1) - y^T), y(0) = 0, \psi_1(0) = 0, \end{cases}$$

with f given as solution of the minimization dual problem

$$\inf_{f \in L^2(\Omega)} \left\{ \frac{1}{2} \int_{\mathcal{O} \times (0, T)} |\varphi|^2 dxdt + \delta \|\widehat{f}\|_{L^2(\Omega)} - \int_{\Omega} f y^T dx \right\}.$$

ii) *The minimization dual problem has a unique solution.*

PROOF. i) Let

$$F(v) = \frac{1}{2} \int_{\mathcal{O} \times (0, T)} |v|^2 dxdt$$

and

$$G(f) = \begin{cases} 0 & \text{if } f \in y^T + \delta B_{L^2(\Omega)} \\ +\infty & \text{otherwise on } L^2(\Omega) \end{cases}$$

Then, an equivalent formulation is

$$\inf_{v \in L^2(\mathcal{O} \times (0, T))} (F(v) + G(Lv))$$

where $Lv = y(T; v)$. By Fenchel and Rockafellar’s (see e.g. Rockafellar [33]) duality

$$\inf_{v \in L^2(\mathcal{O} \times (0, T))} (F(v) + G(Lv)) = - \inf_{f \in L^2(\Omega)} (F^*(L^* f) + G^*(-f)),$$

where L^* is the adjoint operator and F^* the conjugate function

$$F^*(\varphi) = \sup_{\widehat{\varphi}} ((\varphi, \widehat{\varphi}) - F(\widehat{\varphi})).$$

But it is easy to check that

$$F^*(\varphi) = \varphi, G^*(f) = \delta \|\widehat{f}\|_{L^2(\Omega)} + \int_{\Omega} f y^T dx, \text{ and } L^* f = \varphi\chi$$

which gives conclusion i). The proof of ii) follows from some well-known arguments (see [31], [14] and [12]) and comes from the fact that $I(f)$ is strictly convex, continuous and coercive (by the *unique continuation theorem*: see [14] and [12]). ■

Remark 7 As in Lions [24], f is characterized as the unique solution of the *Variational Inequality*

$$(y(T; f) - y^T, \widehat{f} - f) + \delta \|\widehat{f}\|_{L^2(\Omega)} - \delta \|f\|_{L^2(\Omega)} \geq 0, \forall \widehat{f} \in L^2(\Omega). \quad \blacksquare$$

Remark 8 The approximate controllability result seems to be true for some *non linear state equations* when operator A is given by

$$Ay = -Ly + f(y), \text{ or } Ay = -Ly + \operatorname{div} \mathbf{f}(y),$$

with f (respectively \mathbf{f}) sublinear at the infinity: i.e.

$$|f(s)| \leq C_1 + C_2 |s|, \forall s \in \mathbb{R}, |s| > \overline{M}$$

(with analogous condition for \mathbf{f}). Some different methods can be applied to show this type of results: *controllability via linearisation and fixed point arguments* for the state equation (see [21], [14] and [12]). Another possibility is to show the controllability via some *penalized optimal control problem* as, for instance, the associated to the functional

$$J_k(v) = \frac{1}{2} \|v\|_{L^2(\mathcal{O} \times (0, T))}^2 + \frac{k}{2} \|y(T; v) - y^T\|_{L^2(\Omega)}$$

and passing to the limit, as k increases to infinity. This idea due to Lions [23] can be also applied in [15]. ■

Remark 9 In the case of non linear state equations with a superlinear term (e.g. $f(s) = \lambda |s|^{m-1} s$ with $m > 1$) it may arise the, so called, *obstruction phenomenon*, implying the non approximate controllability for general y^T . This was shown by A. Bamberger (see [21]) by means of an energy method. Another proof can be found by means of the construction of universal super and subsolutions over the exterior to the control subdomains (see [7] and [12]). This kind of technique applies to the Burger equation (see [8]) and is inspired in the pioneering work by Brezis and Lieb [3]. ■

Remark 10 Some related numerical experiences can be found in [13]. In particular, it is illustrated there how *the cost of control decreases with complexity* (a philosophy which can be shown rigorously in some cases: see [10]). ■

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