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# The waiting time property for parabolic problems trough the nondiffusion of support for the stationary problems

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**Abstract.** In this note we study the *waiting time phenomenon* for *local* solutions of the nonlinear diffusion equation through its connection with the *nondiffusion of the support property* for local solutions of the family of discretized elliptic problems. We show that, under a suitable growth condition on the initial datum near the boundary of its support, a finite part of the family of solutions of the discretized problem maintain unchanged its support.

**Propiedad de tiempo de espera a través de la no difusión del soporte para los problemas elípticos discretizados.**

**Resumen.** En esta nota estudiamos el fenónemo de tiempo de espera para soluciones locales de la ecuación de difusión no lineal a través de su conexión con la propiedad de no difusión del soporte para soluciones locales de la ecuación elíptica resultante de la discretización implícita. Mostramos que, bajo una adecuada condición de crecimiento del dato inicial cerca de la frontera de su soporte, una familia finita de las soluciones discretizadas mantiene su soporte fijo.

## 1. Introduction

In this note, we consider *bounded local weak solutions* $u \in C([0,+\infty) : L^1(\Omega)) \cap L^\infty(\Omega \times (0,+\infty))$ of the nonlinear diffusion (sometimes called as *porous media*) equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(|u|^{m-1}u) \text{ and } |u| \leq M & \text{in } \Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open set (not necessarily bounded), $m > 1$, $M > 0$ and $u_0(x)$ is a bounded function. For the sake of the exposition we shall assume that $u$ and $u_0$ are nonnegative functions. Thus $|u|^{m-1}u = u^m$. Notice that we are not specifying any boundary condition on $\partial\Omega \times (0,+\infty)$. Some surveys containing many references on the existence of solutions for the different boundary value problems associated to (1) are [8], [6] and [3], among many others.

It is well-known that the assumption $m > 1$ implies that the equation becomes degenerate (*i.e.* non uniformly parabolic) and that one of the many consequences of this fact is the *finite speed of propagation* property: if the support of $u_0$ is a compact set strictly contained in $\Omega$ then the same occurs for $u(\cdot,t)$, at

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least for any $t > 0$ small enough. A sharper property concerns the, so called, *waiting time property* typical of flat initial data near the boundary of its support: if the initial datum $u_0(x)$ satisfies that

$$u_0(x) \leq C_0 \, |x - x_0|^{\frac{2}{m-1}} \quad \text{a.e. } x \in \Omega \ \text{ with } |x - x_0| < \delta, \tag{2}$$

for some $x_0 \in \partial\text{supp}\, u_0$, and for some positive constants $C_0$ and $\delta$, then there exists a *waiting time* $t^* > 0$ such that $x_0 \in \partial\text{supp}\, u(\cdot, t)$ for any $t \in [0, t^*]$. So, if, for instance, $u$ is continuous, then $u(x_0, t) = 0$ for $0 \leq t \leq t^*$.

The main goal of this note is to prove that the waiting time property also holds for the associated discrete solutions (and, in fact it can be proved by passing to the limit) when we consider (as for the existence theory via *accretive operators*) the implicit discretizated in time scheme

$$\frac{u_n - u_{n-1}}{\tau} = \Delta u_n^m \quad \text{and } |u_n| \leq M \quad \text{in} \quad \Omega. \tag{3}$$

Notice that, again, we are not specifying any boundary condition on $\partial\Omega$. We assume that the functions $u_n(x)$ appearing in this iterative scheme represent an approximation of the solution $u(x, t)$ at times $t_n = n\tau$. We assume that $w(x) := u_n(x)^m$ are *nonnegative local bounded weak solutions* of the semilinear elliptic equation

$$-\Delta w + \frac{1}{\tau} w^{1/m} = f(x) \text{ in } \Omega, \tag{4}$$

where $f := \frac{u_{n-1}}{\tau}$. The compactness of the support of the solutions of (4), assumed $f(x)$ with compact support and $\tau$ small enough, is, again, a consequence of the assumption $m > 1$ (see [4] and many other references in the monograph [5]). In fact it was proved in [5], by first time, that under suitable assumptions there is the, so called, *nondiffusion of the support property* ensuring that the support of $f$ coincides with the support of the solution $w$ of (4). The optimality of such assumptions was proved in [2] (see also [1]). For the connection between other qualitative properties of solutions and their version for the solutions of some discretized algorithms see [7].

The main result we present here is the following

**Theorem 1** *Let $u_0(x)$ be a bounded non-negative function satisfying the growth condition (2) for some $x_0 \in \Omega$ and let $\{u_n(x)\}_{n\in\mathbb{N}}$ be any sequence of local bounded nonnegative solutions of (3). Then there exists $C_n$, $\tau_0$ and $t^* > 0$ such that if $0 < \tau < \tau_0$ we have that for any $n \in [0, \frac{t^*}{\tau}) \cap \mathbb{N}$*

$$u_n(x) \leq C_n \, |x - x_0|^{\frac{2}{m-1}} \quad \textit{a.e. } x \in \Omega \ \textit{ with } |x - x_0| < \delta. \tag{5}$$

**Remark 1** It is well known ([5], [8]) that by the maximum principle we have that $\text{supp}\, u_0 \subset \text{supp}\, u_n$. Then the above result proves that, in that case, if (2) holds for any $x_0 \in \partial\, \text{supp}\, u_0$ then $\text{supp}\, u_0 = \text{supp}\, u_n$ for any $n \in [0, \frac{t^*}{\tau}) \cap \mathbb{N}$. ■

## 2. Proof of Theorem 1

We need the following technical lemma

**Lemma 1** *Given $m > 1$, define the function $\phi(s) = s - s^m$ for any $s \in (0, (\frac{1}{m})^{\frac{1}{m-1}}]$. Then*

$$\lim_{n\to\infty} \phi^n(s) n^{\frac{1}{m-1}} = \left(\frac{1}{m-1}\right)^{\frac{1}{m-1}},$$

*where* $\phi^n(s) = \overbrace{\phi \circ \phi \circ \ldots\ldots \circ \phi}^{n}(s)$.

PROOF. First we note that $\phi(s)$ is an increasing function on the interval $\left(0, \left(\frac{1}{m}\right)^{\frac{1}{m-1}}\right]$ and that $\{\phi^n(s)\}_{n\in N}$ is a decreasing sequence converging to 0 (0 is a fixed point of $\phi(s)$) when $n$ goes to infinity. We observe that for any $k \in \mathbb{N}$

$$\limsup_{n\to\infty} \phi^n(s)(n)^{\frac{1}{m-1}} = \limsup_{n\to\infty} \phi^{n+k}(s) n^{\frac{1}{m-1}} = \limsup_{n\to\infty} \phi^n(s)(n+k)^{\frac{1}{m-1}}$$
$$\liminf_{n\to\infty} \phi^n(s)(n)^{\frac{1}{m-1}} = \liminf_{n\to\infty} \phi^{n+k}(s) n^{\frac{1}{m-1}} = \liminf_{n\to\infty} \phi^n(s)(n+k)^{\frac{1}{m-1}}.$$

Let us show that for any $C > \left(\frac{1}{m-1}\right)^{\frac{1}{m-1}}$ and $s \in \left(0, \left(\frac{1}{m}\right)^{\frac{1}{m-1}}\right]$

$$\limsup_{n\to\infty} \phi^n(s)(n)^{\frac{1}{m-1}} < C. \tag{6}$$

Indeed, let $s \in \left(0, \left(\frac{1}{m}\right)^{\frac{1}{m-1}}\right]$ and let $n_0 \in \mathbb{N}$ be such that $C n_0^{-\frac{1}{m-1}} \leq \left(\frac{1}{m}\right)^{\frac{1}{m-1}}$ and satisfying

$$C^{m-1} \geq \frac{n^{-\frac{1}{m-1}} - (n+1)^{-\frac{1}{m-1}}}{n^{-\frac{m}{m-1}}} \quad \text{for any } n \geq n_0. \tag{7}$$

We note that such a $n_0$ always exists since

$$\lim_{n\to\infty} \frac{n^{-\frac{1}{m-1}} - (n+1)^{-\frac{1}{m-1}}}{n^{-\frac{m}{m-1}}} = \frac{1}{m-1} < C^{m-1}. \tag{8}$$

Let $k \in \mathbb{N}$ be such that $\phi^{n_0+k}(s) \leq C n_0^{-\frac{1}{m-1}}$. Let us prove, by induction, that for any $n \geq n_0$

$$\phi^{n+k}(s) \leq C n^{-\frac{1}{m-1}}. \tag{9}$$

Suppose that (9) is verified by $n \geq n_0$, then using (7) we obtain

$$\phi^{n+1+k}(s) \leq \phi(C n^{-\frac{1}{m-1}}) = C n^{-\frac{1}{m-1}} - C^m n^{-\frac{m}{m-1}} \leq C(n+1)^{-\frac{1}{m-1}},$$

which proves estimate (9) for $n \geq n_0$. Therefore

$$\limsup_{n\to\infty} \phi^n(s)(n)^{\frac{1}{m-1}} = \lim\ sup_{n\to\infty} \phi^{n+K}(s) n^{\frac{1}{m-1}} \leq C,$$

for any $C > \left(\frac{1}{m-1}\right)^{\frac{1}{m-1}}$ and we conclude that

$$\limsup_{n\to\infty} \phi^n(s)(n)^{\frac{1}{m-1}} \leq \left(\frac{1}{m-1}\right)^{\frac{1}{m-1}}. \tag{10}$$

In order to prove the reverse inequality, let $C < \left(\frac{1}{m-1}\right)^{\frac{1}{m-1}}$, $s \in \left(0, \left(\frac{1}{m}\right)^{\frac{1}{m-1}}\right]$ and $n_0 \in \mathbb{N}$ be such that $C n_0^{-\frac{1}{m-1}} \leq \left(\frac{1}{m}\right)^{\frac{1}{m-1}}$ and satisfying (7). Let $k \in \mathbb{N}$ be such that $\phi^{n_0}(s) \geq C(n_0+k)^{-\frac{1}{m-1}}$. Let us prove, by induction, that for any $n \geq n_0$

$$\phi^n(s) \geq C(n+k)^{-\frac{1}{m-1}}. \tag{11}$$

Suppose that (11) is verified by $n \geq n_0$, then using (7) we get

$$\phi^{n+1}(s) \geq \phi(C(n+k)^{-\frac{1}{m-1}}) = C(n+k)^{-\frac{1}{m-1}} - C^m(n+k)^{-\frac{m}{m-1}} \geq C(n+1+k)^{-\frac{1}{m-1}}$$

which proves estimate (11) for $n \geq n_0$. Therefore

$$\liminf_{n\to\infty} \phi^n(s)(n+k)^{\frac{1}{m-1}} = \liminf_{n\to\infty} \phi^n(s) n^{\frac{1}{m-1}} \geq C,$$

for any $C < \left(\frac{1}{m-1}\right)^{\frac{1}{m-1}}$ and we obtain

$$\liminf_{n\to\infty}\phi^n(s)n^{\frac{1}{m-1}} \geq \left(\frac{1}{m-1}\right)^{\frac{1}{m-1}}. \tag{12}$$

This concludes the proof by taking into account inequalities (10) and (12). ■

PROOF OF THEOREM 1. Let $\tau > 0$, and let $u_0$ be a non-negative bounded function satisfying the growth condition (2). Without loss of generality we can assume that $x_0 = 0$. We note that the semilinear elliptic equation (3) applied to a radially symmetric function can be written as

$$\mathcal{L}(u_n)(r) = \left[u_n - \tau\left((u_n^m)'' - \frac{N-1}{r}(u_n^m)'\right)\right]r = u_{n-1}(r). \tag{13}$$

Given $k, C > 0$, we consider the *barrier* function

$$v(r) = kC\left(\frac{r^2}{\tau}\right)^{\frac{1}{m-1}}. \tag{14}$$

Let $\{u_n(x)\}$ be any local bounded nonnegative solution of (3) and let $B(0,\delta) \subset \Omega$, be the ball of radius $\delta$ and center $x = 0$. Let us prove that there exist $\tau_0, t^*, k, C > 0$ such that if $0 < \tau < \tau_0$, and $n \leq \frac{t^*}{\tau}$ then

$$u_n(x) \leq v(|x|) \quad \text{in } B(0,\delta). \tag{15}$$

We first notice that an straightforward (but tedious) computation yields to

$$\mathcal{L}(v)(r) = \left(Ck - C^m k^m \frac{m(4m - 2N(m-1))}{(m-1)^2}\right)\left(\frac{r^2}{\tau}\right)^{\frac{1}{m-1}}. \tag{16}$$

Let us start by considering the case $4m - 2N(m-1) \leq 0$. We point out that we have that $|u_n(x)| \leq M$. We take now $k = 1$, $\tau_0 < 1$ and $t^* > 0$ such that

$$C_0\left(\frac{\delta^2}{\tau_0}\right)^{\frac{1}{m-1}} \geq M, \tag{17}$$

where $C_0, \delta$ are defined in the growth condition (2). Then following (2), (16) and (17), we obtain

$$\begin{cases} \mathcal{L}(u_1) \leq \mathcal{L}(v) & \text{in } B(0,\delta), \\ u_1 \leq v & \text{on } \partial B(0,\delta), \end{cases} \tag{18}$$

and therefore by the maximum principle we have that $u_1 \leq v$ in $B(0,\delta)$. Now, by induction on $n$, we obtain (15) which concludes the proof of the theorem in this case.
Let us now consider the case $4m - 2N(m-1) > 0$. If we define

$$k = k_0 = \left(\frac{(m-1)^2}{4m^2 - 2mN(m-1)}\right)^{\frac{1}{m-1}},$$

we obtain, following (16), that

$$\mathcal{L}(v)(r) = k_0\phi(C)\left(\frac{r^2}{\tau}\right)^{\frac{1}{m-1}},$$

where $\phi(s) = s - s^m$. Therefore, the iterations of operator $\mathcal{L}(v)$ yield to

$$\mathcal{L}^n(v)(r) = k_0\phi^n(C)\left(\frac{r^2}{\tau}\right)^{\frac{1}{m-1}}.$$

Now, given $t > 0$ and $n = E(\frac{t}{\tau})$ (where $E(s)$ is the greatest integer satisfying $E(s) \leq s$), from the previous lemma we obtain that if $0 < C \leq \left(\frac{1}{m}\right)^{\frac{1}{m-1}}$

$$\lim_{\tau\to 0^+} \mathcal{L}^n(v)(r) = \lim_{\tau\to 0^+} k_0\phi^n(C)\left(\frac{r^2}{\tau}\right)^{\frac{1}{m-1}} = \lim_{\tau\to 0^+} k_0\left(\frac{1}{m-1}\right)^{\frac{1}{m-1}}\left(\frac{\tau}{t}\right)^{\frac{1}{m-1}}\left(\frac{r^2}{\tau}\right)^{\frac{1}{m-1}} \tag{19}$$

$$= k_0\left(\frac{1}{m-1}\right)^{\frac{1}{m-1}}\left(\frac{r^2}{t}\right)^{\frac{1}{m-1}}, \tag{20}$$

since $\phi^n(C)$ is a decreasing sequence. According (19) we can choose $\tau_0 > 0$ and $t = t^*$ such that if $0 < \tau < \tau_0$ and $n < \frac{t^*}{\tau}$

$$C_0\left(\frac{r^2}{\tau_0}\right)^{\frac{1}{m-1}} \leq \mathcal{L}^n(v) \leq \mathcal{L}^{n-1}(v) \leq \cdots \leq v$$

$$M < k_0\left(\frac{1}{m-1}\right)^{\frac{1}{m-1}}\left(\frac{\delta^2}{t^*}\right)^{\frac{1}{m-1}}.$$

Then, by induction and using the maximum principle as in the previous case, we get that

$$\begin{cases} \mathcal{L}(u_n) \leq \mathcal{L}(v) & \text{in } B(0,\delta) \\ u_n \leq v & \text{on } \partial B(0,\delta), \end{cases} \tag{21}$$

and therefore (15) is shown. This concludes the proof of Theorem 1. ∎

**Remark 2** The above theorem is, in some sense, optimal, since in the case of general non-linearities $\varphi(u)$ in the nonlinear diffusion equation the waiting time property is not always preserved when we discretize in time the equation. To explain it and to fix ideas, let us consider local nonnegative solutions of the non-linear degenerate parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta\varphi(u) & \text{in} \quad \Omega\times(0,+\infty), \\ u(x,0) = u_0(x) & \text{on} \quad \Omega, \end{cases} \tag{22}$$

where $\varphi$ is a continuous nondecreasing function such that $\varphi(0) = 0$, and let $\{u_n(x)\}_{n\in\mathbb{N}}$ be any sequence of local bounded nonnegative solutions of the associated semidiscrete iterative elliptic equation

$$\frac{u_n - u_{n-1}}{\tau} = \Delta\varphi(u_n) \quad \text{in} \quad \Omega. \tag{23}$$

It is well known (see references in [8], [3] or [6]) that equation (22) admits local solutions $u(\cdot,t)$ with compact support solutions, at least for $t$ small enough, iff

$$\int_{0^+} \frac{ds}{\varphi^{-1}(s)} < \infty \tag{24}$$

(notice that $\varphi^{-1}(s)$ is well defined, as a maximal monotone graph and as a function *a.e.* $s \in \varphi(\mathbb{R})$). On the other hand, equation (23) admits local solutions with compact support iff

$$\int_{0^+} \frac{1}{\sqrt{F(s)}}ds < \infty, \tag{25}$$

where $F(s)$ is defined as $F(s) = \int_0^s \varphi^{-1}(z)dz$ (see [5] and its references). Finally, it is easy to see that, although both conditions are equivalent in the case $\varphi(s) = s^m$ once that $m > 1$, condition (24) is stronger than (25): that is (25) implies (24) but there are nonlinearities $\varphi(s)$ which satisfy (24) but not (25): take, for instance, $\varphi(s) = s\ln s(1+\ln s)$. In this way, we can not expect to prove the waiting time property (for a general function $\varphi(s)$ satisfying (24) through the nondiffusion of the support property of the associated discretized family of elliptic problems. ∎

**Remark 3** Theorem 1 can be easily extended to the, so called, doubly nonlinear problems of the type

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_p(|u|^{m-1}u) & \text{in} \quad \Omega \times (0, +\infty), \\ u(x,0) = u_0(x) & \text{on} \quad \Omega, \end{cases} \tag{26}$$

with $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ once we assume $m > 0$, $p > 1$ and, which is crucial, $m(p-1) > 1$. Many references on this equation are given in the surveys [8], [3] or [6]. The consideration of the associated stationary problems

$$\frac{u_n - u_{n-1}}{\tau} = \Delta_p u_n^m \quad \text{in} \quad \Omega, \tag{27}$$

was carried out in the monograph [5]. For instance, $v(r) = kC\left(\frac{r^p}{\tau}\right)^{\frac{1}{(p-1)m-1}}$ is now the barrier function. ▩

**Remark 4** Theorem 1 also applies to changing sign solutions of any boundary value associated to the equation since it is enough to apply the barrier comparison functions (and the corresponding iteration process) to the one sign solutions corresponding to the positive and negative parts of the initial datum (this argument is standard in this type of equations for which the maximum principle holds: see the references mentioned in the above Remark). ▩

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