myjournal manuscript No.
(will be inserted by the editor)

On an elliptic system related to desertification studies

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Received: date / Accepted: date

Abstract In this communication, we consider the stationary problem of a non-linear parabolic system which arises in the context of dry-land vegetation. In the first part, we examine the existence and multiplicity of biomass stationary solutions, in terms of the precipitation rate parameter p, for a localized simplification of the system, with non-homogeneous rate of biomass loss. In fact, we show that under appropriate conditions on fixed parameters of the problem, multiple positive solutions exist for a range of the parameter p. In the second part, we consider the case of an idealized "oasis", $\omega \subset\subset \Omega$, where we study the transition of the surface-water height in a neighborhood of the set ω.

Keywords Multiplicity of positive solutions · Dry-land vegetation

1 Introduction

We study a system of elliptic equations which is the stationary version of a dry-land vegetation model proposed by Gilad et al. (2007). Precisely, the stationary problem is given by the following elliptic system:

$$\begin{cases} -\delta_b \Delta b = -\mu b + G_b b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta w = -G_w w - \mathscr{E}_b w + \mathscr{I}_b h & \text{in } \Omega, \\ -\delta_h \Delta h^2 = -\mathscr{I}_b h + p & \text{in } \Omega, \\ \dfrac{\partial b}{\partial n} = \dfrac{\partial w}{\partial n} = \dfrac{\partial h}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where we suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2 boundary and n denotes the outward pointing unit normal vector field on $\partial\Omega$. Here, b represents the biomass, w the soil-water content and h the surface-water height after suitable non-dimensionalization. The growth rate G_b and the water uptake rate G_w are non-local terms given by

$$G_b(b,w) = \nu \int_\Omega g(x,y)w(y)\,dy \quad \text{and} \quad G_w(b) = \gamma \int_\Omega g(y,x)b(y)\,dy,$$

where $g(x,y) = \dfrac{1}{2\pi\sigma^2}\exp\big[-\dfrac{|x-y|^2}{2(\sigma(1+\eta b))^2}\big]$ for $x,y \in \Omega$. Moreover, $\mu > 0$ represents the biomass loss rate, $\mathscr{E}_b(b) = \dfrac{\nu}{1+\rho b}$ is the evaporation rate of the soil water, and $\mathscr{I}_b(b) = \alpha\dfrac{b+q/c}{b+q}$ represents the infiltration rate of the surface-water. The third equation of the parabolic system is a porous medium type equation and involves the precipitation rate parameter $p > 0$. The rest of the parameters are positive, and in fact, $c > 1$. As we shall see, in special cases, some of the parameters may also be taken to be equal to zero. (More information about the modeling may be found in Gilad et al. (2007); Meron (2011)).

In section 1, we consider the case of plant species with negligible below-ground biomass. In that case we may assume that the root extension parameter η is equal to zero. Furthermore, since the minimal root size of such plant species tends to zero, the non-local effect of the root system is insignificant. In particular, we may replace $g(x,y)$ with the Dirac delta based on x, arriving at the following local coupled system

$$\begin{cases} -\delta_b \Delta b = -\mu b + \nu w b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta w = -\gamma b w - \mathscr{E}_b w + \mathscr{I}_b h & \text{in } \Omega, \\ -\delta_h \Delta h^2 = -\mathscr{I}_b h + p & \text{in } \Omega, \\ \dfrac{\partial b}{\partial n} = \dfrac{\partial w}{\partial n} = \dfrac{\partial h}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Moreover, we shall limit ourselves to the case where infiltration feedback and soil-water diffusion are not present, which corresponds to $\delta_w = \delta_h = 0$. Finally, we consider an inhomogeneous biomass loss rate μ, which cannot exceed a minimal loss rate due to natural mortality and a maximal total loss rate. Precisely, in dimensionless quantities, we suppose that $\mu \in C^1(\bar{\Omega})$ is such that $1 \le \mu(x) \le \bar{\mu}$, for $x \in \bar{\Omega}$. On the basis of the considerations described above, in the first section we shall investigate the existence of positive solutions in terms of the precipitation parameter p.

In the third part, we consider system (1) assuming that the precipitation rate is inhomogeneous. Particularly, we assume that p is constant in a closed subset $\omega \subset\subset \Omega$ and vanishes outside ω. Here, one may think of $p(\cdot)$ as a distributed water resource which is not negligible only on a sub-region ω of Ω. Moreover, in contrast with section 2, in this section we suppose that $\delta_w, \delta_h > 0$ and that the loss rate μ is a positive constant. In that occasion, we investigate the free boundary of the surface-water solution component h, in terms of the parameters involved in the third equation of (1).

2 A multiplicity result

In this section, we seek non-negative solutions of (2), depending on the parameter p, when $\delta_w = \delta_h = 0$ and $\mu(x)$ is a smooth function in Ω such that $1 \leq \mu(x) \leq \bar{\mu}$. In fact, without loss of generality we also assume that $\delta_b = 1$. Thus, we study the following elliptic problem

$$\begin{cases} -\Delta b + \mu(x) b = p f(b) & \text{in } \Omega, \\ \dfrac{\partial b}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{P}_p$}$$

for

$$f(b) = \frac{\nu b(1-b)(1+\rho b)}{\gamma b(1+\rho b)+\nu}.$$

Clearly, $f(\cdot) \in C^2(\mathbb{R}^+)$,

$$0 \leq f(s) \leq M \quad \text{for all} \quad s \in [0,1],$$

and

$$f(s) < 0 \quad \text{for} \quad s \geq 1.$$

We also note, that $b \equiv 0$ is a solution of ($\mathscr{P}_p$) for all $p > 0$, such a solution will be called the trivial. We shall first consider a subclass of weak solutions, namely, the so-called variational solutions. So, let us consider the set

$$K = \{v \in H^1(\Omega) \,|\, 0 \leq v \leq 1 \,\text{in}\, \Omega\},$$

and let

$$F_p(v) = p \int_0^v f(s)\, ds\,.$$

We define the variational functional

$$J_p(v) = \frac{1}{2} \int_\Omega \left(|\nabla v|^2 + \mu(x) v^2\right) dx - \Phi_p(v),$$

where

$$\Phi_p(v) := \int_\Omega F_p(v(x))\, dx.$$

Definition 1 We shall call a function $v \in H^1(\Omega)$, a variational solution of ($\mathscr{P}_p$), if v is a minimum of the functional J_p on the set K.

Remark 1 It can be easily verified that any variational solution is a weak solution (it suffices to consider the Euler-Lagrange equation associated to the functional J_p).

We have:

Theorem 1 *For each $p > 0$, there exists at least one variational solution of ($\mathscr{P}_p$).*

Proof Since K is a convex closed subset of $H^1(\Omega)$, in order to show that J_p attains a minimum (due to a version of the Weierstrass theorem, see Zeidler (1990) p.513), it suffices to show that J_p is weakly lower semicontinuous and weakly coercive defined on K.

(i) J_p is weakly lower semicontinuous. Indeed, the norm on $H^1(\Omega)$ is weakly lower semicontinuous. On the other hand, the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $1 \leq q < \infty$, since $N = \dim(\Omega) = 2$. Therefore, if v_n is a sequence in K that converges weakly in $H^1(\Omega)$ to a function v, we know that (up to a subsequence) $v_n \to v$ strongly in $L^q(\Omega)$. This actually implies that

$$\Phi_p(v_n) \to \Phi_p(v)$$

and so the map $\Phi_p : K \subset H^1(\Omega) \to R$ is weakly continuous. Thus $J_p(v)$ is weakly lower semicontinuous.

(ii) J_p is coercive. Indeed, for $u \in K$ we have $\Phi_p(v) \leq pM\|v\|_{L^1(\Omega)} \leq pM|\Omega|$ so for some constant $C(p,\Omega) > 0$, $J_p(v) \geq \dfrac{1}{2}\|v\|^2_{H^1} - C(p,\Omega)$ which implies that $J(v) \to \infty$ as $\|v\|^2_{H^1} \to \infty$. This ends the proof of the Theorem 1.

We now proceed to consider solutions of $(\mathscr{P}_p)$ which are not necessarily variational solutions. Our study is inspired by a previous one arising in a completely different context: some simple climate models Díaz et al. (1997).

Before stating our main result, it is useful to consider the following auxiliary algebraic equations which provide us with positive constant super and sub-solutions of $(\mathscr{P}_p)$ for a range of the parameter p.

$$s = pf(s), \quad s \in \mathbb{R}, \tag{E_1}$$

$$\bar{\mu}s = pf(s), \quad s \in \mathbb{R}. \tag{E_2}$$

So, let us make some observations and introduce notation related to the set of non-negative solution of (E_1) and (E_2). We first observe that for all $p > 0$, $s = 0$ satisfies both equations and any possible positive solution of the auxiliary equations has to be lower than unity. We shall denote by Γ_1 and Γ_2 the (bifurcation) curves of nontrivial positive solutions corresponding to the algebraic equations (E_1) and (E_2), respectively. Now, let

$$T_1(p,s) = s - pf(s)$$

and

$$T_2(p,s) = \bar{\mu}s - pf(s).$$

Clearly, if p_i is such that

$$\frac{\partial}{\partial s}T_i(p_i,0) = 0, \tag{3}$$

then Γ_i bifurcates from the line of trivial solutions at $(p_i,0)$, for $i = 1,2$. One can easily check that $p_1 = 1$ and $p_2 = \bar{\mu}$. Moreover, if the following condition is satisfied

$$\rho > 1 \quad \text{and} \quad \gamma < \nu(\rho - 1), \tag{C1}$$

then $\dfrac{\partial^2}{\partial s^2}T_i(p_i,0) < 0$ and the bifurcation at $(p_i,0)$ is subrcritical. In fact, (C1) also assures that Γ_i has a unique "turning point" (p_i^*,s^*) which satisfies

$$T_i(p_i^*,s^*) = 0 = \frac{\partial}{\partial s}T_i(p_i^*,s^*) \tag{4}$$

as well as

$$\frac{\partial^2}{\partial s^2}T_i(p_i^*,s^*)>0 \text{ and } \frac{\partial}{\partial p}T_i(p_i^*,s^*)<0,$$

for $i=1,2$, where

$$s^*=\frac{-(\gamma+\nu)+\sqrt{\nu^2+\gamma\nu+\rho\gamma\nu}}{\gamma\rho}>0.$$

Finally, we point out that for fixed $p\in(p_i^*,p_i)$, (E_i) has two distinct positive solutions denoted by $s_{i,p}^1$ and $s_{i,p}^2$, which are such that

$$s_{i,p}^1<s^*<s_{i,p}^2,$$

while for $p\geq p_i$, (E_i) has a unique positive solution denoted again by $s_{i,p}^2$.

Theorem 2 *Let p_1,p_2 be the bifurcation points of (E_1), (E_2) given by (3). Also, assume that (C1) holds true and let p_i^* be the unique points that satisfy (4) for $i=1,2$. Then,*

(i) if $p\in(0,p_1^*)$, the trivial solution $b\equiv 0$ is the only possible non-negative solution of $(\mathscr{P}_p)$.
(ii) if $p_2^*<p_1$ and $p\in(p_2^*,p_1)$, $(\mathscr{P}_p)$ has at least two positive solutions, besides the trivial solution $b\equiv 0$.
(iii) if $p\in(\max\{p_1,p_2^*\},\infty)$, then besides the trivial solution, $(\mathscr{P}_p)$ has at least one positive solution. In fact, for p large enough, there exists $\xi\in(0,1)$ and a unique non-trivial positive solution of $(\mathscr{P}_p)$ satisfying $\xi\leq b(x)<1$ in Ω. Moreover, this unique solution is also a variational solution of $(\mathscr{P}_p)$.

Proof (i) By $(C1)$, there exists a unique pair $(p_1^*,s^*)\in\Gamma_1$ such that $f'(s^*)>0$. In fact, we have that

$$f(s)\leq sf'(s^*) \quad \text{for all} \quad s\geq 0. \tag{5}$$

Therefore, if $b\in H^1(\Omega)$ is a non-negative solution of $(\mathscr{P}_p)$, by multiplying $(\mathscr{P}_p)$ by b and integrating over Ω, since $\mu(x)\geq 1$, we have that

$$\int_\Omega b^2\,dx\leq\int_\Omega\left(|\nabla b|^2+b^2\right)dx\leq\int_\Omega f(b)b\,dx\leq pf'(s^*)\int_\Omega b^2\,dx.$$

So, a non-negative solution which is not the trivial may exist only for $p>p_1^*$. In order to obtain (ii) and (iii), we now focus on positive constant super and sub-solutions of $(\mathscr{P}_p)$. Clearly, for $p>0$ any positive solutions of the following problems

$$\begin{cases}-\Delta U_p+U_p=pf(U_p) & \text{in } \Omega,\\ \frac{\partial U_p}{\partial n}\geq 0 & \text{on } \partial\Omega,\end{cases}$$

and

$$\begin{cases}-\Delta V_p+\bar{\mu}V_p=pf(V_p) & \text{in } \Omega,\\ \frac{\partial V_p}{\partial n}\leq 0 & \text{on } \partial\Omega,\end{cases}$$

are respectively, sup or sub-solutions of $(\mathscr{P}_p)$. So, for every $p>p_1^*$ positive solutions of (E_1) form a family of positive constant super-solutions and for every $p>p_2^*>0$ positive solutions of (E_2) form a family of sub-solutions of $(\mathscr{P}_p)$. Namely, we let

$$U_p^1\equiv s_{1,p}^1,\ U_p^2\equiv s_{1,p}^2 \text{ and } V_p^2\equiv s_{2,p}^2.$$

(ii) If $p_2^* < p_1$, for each $p \in (p_2^*, p_1)$, we consider the ordered intervals $[0, U_p^1]$ and $[V_p^2, U_p^2]$, then from (Theorem 15.2 Amann 1976, p.668), $(\mathscr{P}_p)$ has at least three distinct solutions b_1, b_2 and b_3 such that $0 \leq b_1 < b_2 < b_3 < U_p^2$. Since b_1 may be identically zero (ii) follows.

(iii) For each $p \in (\max\{p_1, p_2^*\}, \infty)$, we consider the ordered interval $[V_p^2, U_p^2]$ where V_p and U_p are such that $V_p < U_p$. It is easy to check that the conditions of the results in Amann (1976) hold true and so there exist a minimal and a maximal solution in $[V_p^2, U_p^2]$. Moreover, for p large enough, any such positive weak solution takes values in an interval $[\xi, 1)$ where $f(\cdot)$ is decreasing which implies the uniqueness of any possible weak solution taking values in that interval. Finally, the energy of such weak solution is less than zero. Therefore, we deduce that for p large enough u_p is also a variational solution of $(\mathscr{P}_p)$.

It is actually natural to ask whether the set of positive solutions consists of a connected closed set in $R \times X$ for some function space X. To this end, we let $X = C(\bar{\Omega})$ and recall that X possesses a positive cone P induced by the natural ordering. In fact, since P has non-empty interior, P is total i.e., $\{u - v : u, v \in P - 0\}$ is dense in X. We denote by K the solution operator of $-\Delta + \mu(x)$ together with the homogeneous Neumann boundary conditions, and by $F : P \to X$ the Nemiskii operator given by $F(u) = f^+(u(\cdot))$. Note that $f^+ \in C^{0,\alpha}(R)$ and F is continuous and clearly, if $u \in P$, then $F(u) \in P$. Now let us consider the following auxiliary problem.

$$\begin{cases} -\Delta b + \mu(x) b = p f^+(b) & \text{in } \Omega, \\ \dfrac{\partial b}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \qquad (\mathscr{P}_p^+)$$

Since K is a linear positive compact operator from X to itself, we have that for $p > 0$ the map

$$pK \circ F : P \to X$$

is completely continuous and positive, where the latter means that $pK \circ F(P) \subset P$. Finally, the fixed point equation $u = pK \circ Fu$ for $u \in X$, is equivalent to equation $(\mathscr{P}_p^+)$. It can be checked that $K \circ F$ is right Frechet differentiable at $u = 0$, with K being the Frechet derivative from the right. Therefore by Theorem 18.3 in Amann (1976) (see also Dancer (1973)), we conclude that:

Theorem 3 *The problem $(\mathscr{P}_p^+)$ possesses an unbounded continuum of positive solutions $\mathscr{C}^+$ in $R^+ \times P$, emanating from the line of trivial solutions $(p, 0)$ at $(p^*, 0)$, where p^* is the unique positive eigenvalue with a positive eigenvector of the following eigenvalue problem:*

$$\begin{cases} -\Delta b + \mu(x) b = \lambda b & \text{in } \Omega, \\ \dfrac{\partial b}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \qquad (6)$$

Remark 2 By standard regularity theory and the strong maximum principle, we have that $\max_{\bar{\Omega}} b(x) < 1$. Therefore, since $f(s) = f^+(s)$ for $s \in [0, 1]$, we have that $(p, b) \in \mathscr{C}^+$ also satisfies $(\mathscr{P}_p)$. Note that this is true for all $\nu, \gamma, \rho > 0$.

3 Surface water transition

In this section, we study the third equation of system (1), assuming that $\delta_w, \delta_h > 0$ and that the precipitation rate is not completely constant in Ω, but vanishes outside a closed subset $\omega \subset\subset \Omega$. For $p > 0$, we let $p(x) = p\chi_\omega(x)$ on Ω, where χ_ω denotes the characteristic function of ω. We recall that the non-linear term of the equation involves the so-called infiltration contrast parameter $c > 1$. Now, supposing that b is a given non-negative solution of the corresponding equation of system (1) for the given boundary conditions, we set

$$\theta(x) := \alpha \frac{b(x) + q/c}{b(x) + q} \quad \text{in } \Omega.$$

Obviously, we have that $\frac{\alpha}{c} \leq \theta(x) \leq \alpha$ on Ω. On the other hand, letting $\tilde{h} = h^2$, if $h \geq 0$ and $\delta_h > 0$, then the third equation can be written as

$$\begin{cases} -\Delta \tilde{h} + \dfrac{\theta(x)}{\delta_h}\sqrt{\tilde{h}} = \phi(x) & \text{in } \Omega, \\ \\ \dfrac{\partial \tilde{h}}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

with $\phi(x) := \dfrac{p}{\delta_h}\chi_\omega(x)$.

We point out that, in general, we cannot ensure the uniqueness of function $\tilde{h}$ (in fact, in the preceding section, we exhibit a case of multiplicity of b and so of h). Nevertheless, for fixed b the non-negative solution of (7) is unique. Furthermore, by the maximum principle, we know that a possible solution $\tilde{h}$ must satisfy that

$$\|\tilde{h}\|_{L^\infty(\Omega)} \leq \left(\frac{pc}{\alpha}\right)^2.$$

The following theorem provides an estimate on the location of the null set of a solution component h. This estimate depends on c, α, δ_h and p.

Theorem 4 *Let h be the third component of any non-negative solution of the system (1). Then, necessarily, $h(x) = 0$ for all $x \in \Omega - \omega$ such that*

$$d(x, \partial\Omega \cup \partial\omega) > 4\sqrt{p}\frac{c\sqrt{\delta_h}}{\alpha}.$$

In fact, at least one of those possible solutions verifies that $h(x) = 0$ for any $x \in \Omega - \omega$ such that $d(x, \partial\omega) > 4\sqrt{p}\frac{c\sqrt{\delta_h}}{\alpha}$.

Proof We set $m = \dfrac{\alpha/c}{\delta_h}$. Following Díaz (1985), we look for a local comparison function $\tilde{h}_m$ such that $\tilde{h} \leq \tilde{h}_m$ on the ball $B_R(x_0)$ and $\tilde{h}_m(x_0) = 0$, where $R \geq 4\sqrt{p}\dfrac{c\sqrt{\delta_h}}{\alpha}$ so that $B_R(x_0) \subset \Omega - \omega$. Then, since $\tilde{h} \geq 0$ clearly $\tilde{h}(x_0) = 0$ (and in a weak sense if $\tilde{h}$ is not continuous). In fact, if $\tilde{h}_m \in H^1(\Omega)$ satisfies

$$-\Delta \tilde{h}_m + m\sqrt{\tilde{h}_m} \geq 0 \qquad \text{in} \quad B_R(x_0), \tag{8}$$

$$\tilde{h}_m \geq \left(\frac{pc}{\alpha}\right)^2 \qquad \text{on} \quad \partial B_R(x_0), \tag{9}$$

then, since

$$-\Delta\tilde{h}+m\sqrt{\tilde{h}}=\left(m-\frac{\theta(x)}{\delta_h}\right)\sqrt{\tilde{h}}\leq 0\leq -\Delta\tilde{h}_m+m\sqrt{\tilde{h}_m}\ \text{ in } B_R(x_0),$$

by the comparison principle, we have that $\tilde{h}\leq\tilde{h}_m$.
Now, for such $x_0\in\Omega-\omega$, we consider the function $\tilde{h}_m(x)=C_m|x-x_0|^4$ where $C_m=\left(\frac{m}{16}\right)^2$. Then it is not difficult to check (see Díaz (1985)) that

$$-\Delta\tilde{h}_m+m\sqrt{\tilde{h}_m}\geq 0 \quad \text{in} \quad B_R(x_0),$$

and so the first conclusion holds. The second assertion holds merely by extending by zero some of those solutions on the set of $x\in\Omega-\omega$ such that $d(x,\partial\omega)>4\sqrt{p}\frac{c\sqrt{\delta_h}}{\alpha}$ (since, obviously it also satisfies the Neumann boundary conditions).

Remark 3 In fact, it is possible to give a sharper estimate (near $\partial\Omega$) depending on the geometry of the domain Ω (Díaz 1985, ch. 2) but we shall not detail it here.

Remark 4 From the estimate of the preceding theorem we deduce that the distance of the free boundary from the set ω increases when one of the parameters p, δ_h or c increases or when α decreases. Moreover, the same answer remains true when the variation of the parameters is not necessarily monotone in each of them but the combination of them given by the expression $\frac{\sqrt{p}c\sqrt{\delta_h}}{\alpha}$ increases.

Acknowledgements The research of the authors has received funding from the ITN FIRST of the Seventh Framework Programme of the European Community. (grant agreement number 238702). The research of JID was partially supported by the Research Group MOMAT (Ref. 910480), UCM and the project MTM2011-26119 (DGISPI, Spain).

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