Stability criteria on flat and compactly supported ground states of some non-Lipschitz autonomous semilinear equations

Jesús Ildefonso Díaz, Jesús Hernández and Yavdat Il'yasov *

To a master, Haïm Brezis, with admiration.

Abstract

We analyze the stability of a class of nonnegative ground states, relevant in the applications, of the Dirichlet problem associated to some semilinear autonomous elliptic equations with a strong absorption term given by a non-Lipschitz function on a bounded regular domain Ω of $\mathbb{R}^N$. We prove that ground state (i.e., solutions of minimal energy) with gradient vanishing on the boundary of the domain $\partial\Omega$ and satisfying additionally that either $u > 0$ on Ω (flat solutions) or with support $u \subsetneq \overline{\Omega}$ (compactly supported solutions) are unstable for dimensions $\mathrm{N} = 1, 2$. Flat ground states can be stable for $\mathrm{N} \geq 3$ for suitable values of the involved exponents.

1 Introduction and main results

Let $\mathrm{N} \geq 1$, and let Ω be a bounded domain in $\mathbb{R}^{\mathrm{N}}$ whose boundary $\partial\Omega$ is a $\mathcal{C}^1$-manifold. We consider the following semi-linear parabolic problem

$$PP(\alpha, \beta, \lambda, v_0) \quad \begin{cases} v_t - \Delta v + |v|^{\alpha-1}v = \lambda |v|^{\beta-1}v & \text{in } (0, +\infty) \times \Omega, \\ v = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ v(0, x) = v_0(x) & \text{on } \Omega. \end{cases} \tag{1}$$

Here λ is a positive parameter and $0 < \alpha < \beta \leq 1$. Our main goal is to give some stability criteria on solutions of the associated stationary problem

$$SP(\alpha, \beta, \lambda) \quad \begin{cases} -\Delta u + |u|^{\alpha-1}u = \lambda |u|^{\beta-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Notice that since the diffusion-reaction balance involves the non-linear reaction term

$$f(\lambda, u) := \lambda |u|^{\beta-1}u - |u|^{\alpha-1}u$$

and it is a non-Lipschitz function at zero (since $\alpha < 1$ and $\beta \leq 1$) important peculiar behavior of solutions of both problems arise. For instance, that may lead to the "violation of the Hopf maximum principle" on the boundary so that

$$\frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega. \tag{3}$$

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In fact, the nonnegative solutions of $SP(\alpha, \beta, \lambda)$ satisfying (3) may be classified in two different subclasses: *flat solutions* (sometimes also called *free boundary solutions*) verifying that $u > 0$ on Ω, and *compactly supported solutions,* i.e. such that support $u \subsetneq \overline{\Omega}$.

Solutions of this kind for stationary equations with non-Lipschitz nonlinearity have been investigated in a number of papers. The pioneering paper in which it was proved that the solution gives rise to a free boundary defined as the boundary of its support was due to Haïm Brezis [9] concerning multivalued non-autonomous semilinear equations. The semilinear case with non-Lipschitz perturbations was considered later in [4] (see also [5], [11] and [12]). For the case of semilinear autonomous elliptic equations see e.g. [26], [28], [30], [16], [17], [44], [46], [47], [53], to mention only a few.

For problem (2), the existence of radial flat solutions was first proved by Kaper and Kwong [46]. In this paper, applying shooting methods they showed that there exists $R_0 > 0$ such that (2) considered in the ball $B_{R_0} = \{x \in \mathbb{R}^N : |x| \leq R_0\} = \Omega$ has a radial compactly supported positive solution. Furthermore, by the moving-plane method it was proved in [47] that any classical solution of (2) is necessarily radially symmetric if Ω is a ball. The application of the moving-plane method was later improved by Serrin and Zou [53] by showing that any nonnegative C^1-solution of the equation in $\mathbb{R}^N$ (in fact to a more general class of elliptic equations including $SP(\alpha, \beta, \lambda)$ when $0 < \alpha < \beta \leq 1$) and with compact support must be radially symmetric with respect some origin. In consequence, although it was not indicated in reference [53] we can prove (see Theorems 2 and 3) that *flat* solutions only may exists (and they are radially symmetric) if Ω is a ball. Indeed, any *flat* or *compactly supported solution* may be extended by zero outside generating a classical nonnegative solution of the equation in $\mathbb{R}^N$ with compact support we deduce that any *flat* or *compactly supported solution* must be radially symmetric with respect some origin.

Observe that from this it follows that the Dirichlet boundary value problem (2) has a compactly supported solution if $B_{R_0} \subseteq \Omega$ (see [28] for some more general statements).We point out that for certain elliptic equations containing some transport terms or some non-isotropic coefficients (see, e.g., [22] and [28]) it may exist solutions satisfying support $u \subsetneq \overline{\Omega}$ but not satisfying condition (3) in some part of $\partial\Omega$: those solutions does not generate a classical solution in $\mathbb{R}^N$ when they are extended by zero outside of Ω.

In this work we study the stability of solutions of the stationary problem $SP(\alpha, \beta, \lambda)$. We point out that a direct analysis of the stability of the stationary solutions $u_\infty \in [0, +\infty)$ of the associated ODE

$$ODE(\alpha, \beta, \lambda, v_0) \quad \begin{cases} v_t + |v|^{\alpha-1}v = \lambda |v|^{\beta-1}v & \text{in } (0, +\infty) \\ v(0) = v_0, \end{cases} \tag{4}$$

shows that the trivial solution $u_\infty \equiv 0$ is asymptotically stable and that the nontrivial stationary solution $u_\infty := \lambda^{-1/(\beta-\alpha)}$ is unstable (see Figure 1).

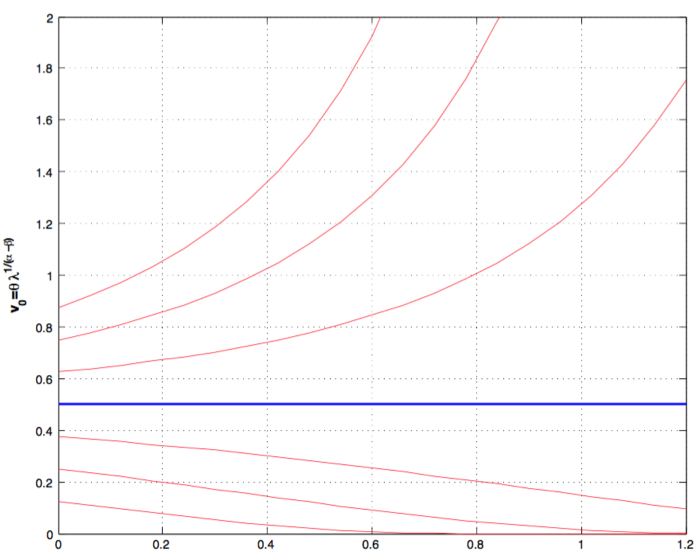


Figure 1: Paths for $\mathrm{ODE}(1/2, 3/2, \lambda, v_0)$.

Obviously the same criteria hold for the case of the semilinear problem with Neumann boundary conditions. Nevertheless, unexpectedly, the situation is not similar for the case of Dirichlet boundary conditions, and so, as the main result of this paper will show, for dimensions $\mathrm{N} \geq 3$ the nontrivial flat solution of $SP(\alpha,\beta,\lambda)$ becomes stable in a certain range of the exponents $\alpha < \beta < 1$. To be more precise, our stability study will concern *ground state* solutions (also called simply *ground states*) of $SP(\alpha,\beta,\lambda)$ and it is inspired on previous stability works devoted (some times in a formal and non rigorous way) to some semilinear hyperbolic problems with different nonlinear terms by G. H. Derrick [21] and L. E. Payne and D. H. Sattinger [50]. In fact, to give a sense to the notion of stability to the class of compactly supported solutions (which usually form a connected set of weak solutions of the equations) we shall introduce in this paper a new notion of stability for such set of solutions which seems to be better adapted to this framework than the particularization of the abstract theory of attractors for general dynamical systems (see, e.g., the survey [35]).

By a *ground state* solution we mean a nonzero weak solution u_λ of $SP(\alpha,\beta,\lambda)$ which satisfies

$$E_\lambda(u_\lambda) \leq E_\lambda(w_\lambda)$$

for any nonzero weak solution w_λ of $SP(\alpha,\beta,\lambda)$. Here $E_\lambda(u)$ is the energy functional corresponding to $SP(\alpha,\beta,\lambda)$ which is defined on the Sobolev space $\mathrm{H}^1_0(\Omega)$ as follows

$$E_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{\alpha+1}\int_\Omega |u|^{\alpha+1}\,dx - \lambda\frac{1}{\beta+1}\int_\Omega |u|^{\beta+1}\,dx.$$

For simplicity, we shall assume the initial value such that $v_0 \in \mathrm{L}^\infty(\Omega)$, $v_0 \geq 0$. As we shall show in Section 2, then there exists a weak solution $v \in \mathrm{C}([0,+\infty);\mathrm{L}^2(\Omega))$ of $PP(\alpha,\beta,\lambda,v_0)$ satisfying $\lambda|v|^{\beta-1}v - |v|^{\alpha-1}v \in \mathrm{L}^\infty((0,+\infty)\times\Omega)$ and

$$v(t) = T(t)v_0 + \int_0^t T(t-s)(\lambda|v|^{\beta-1}v - |v|^{\alpha-1}v)ds, \tag{5}$$

with $(T(t))_{t\geq 0}$ the linear heat semigroup with homogeneous Dirichlet boundary conditions. Among some additional regularity properties of v we mention that

$$v - T(t)v_0 \in \mathrm{L}^p(\tau,T;\mathrm{W}^{2,p}(\Omega)\cap\mathrm{W}^{1,p}_0(\Omega)) \cap \mathrm{W}^{1,p}(\tau,T;\mathrm{L}^p(\Omega)), \tag{6}$$

for every $p \in (1,\infty)$, and for any $0 < \tau < T$ $\big($in fact $\tau = 0$ if we also assume that $v_0 \in \mathrm{W}^{1,p}_0(\Omega)\big)$. In particular, v satisfies the equation $PP(\alpha,\beta,\lambda,v_0)$ for a.e. $t \in (0,+\infty)$. Moreover, if $v(0) \in \mathrm{H}^1_0(\Omega)$ then, for any $t > 0$

$$\int_0^t ||v_t(s)||^2_{L^2}ds + E_\lambda(v(t)) \leq E_\lambda(v(0)). \tag{7}$$

We shall show in Section 2 that there is uniqueness of solutions of $PP(\alpha,\beta,\lambda,v_0)$ in spite of the presence of non-Lipschitz terms in the equation thanks to the crucial assumption $\alpha < \beta$. We also prove that if $\lambda \in [0,\lambda_1)$ then the finite extinction time property is satisfied for solutions of $PP(\alpha,\beta,\lambda,v_0)$ (as in the pioneering paper [13] on multivalued semilinear parabolic problems; see also the survey [23]). Moreover we shall show in Section 2 that there is a certain resemblance between the set of solutions of $PP(\alpha,\beta,\lambda,v_0)$ and the corresponding one of the problem $ODE(\alpha,\beta,\lambda,v_0)$ since: a) for any $\lambda > 0$ the trivial solution $u \equiv 0$ of the stationary problem $SP(\alpha,\beta,\lambda)$ is asymptotically stable in the sense that it attracts solutions of $PP(\alpha,\beta,\lambda,v_0)$ for small initial data v_0 (Proposition 2.1), and b) if v_0 is "large enough" the trajectory of the solution of $PP(\alpha,\beta,\lambda,v_0)$ is unbounded when $t \nearrow +\infty$ (Proposition 2.4).

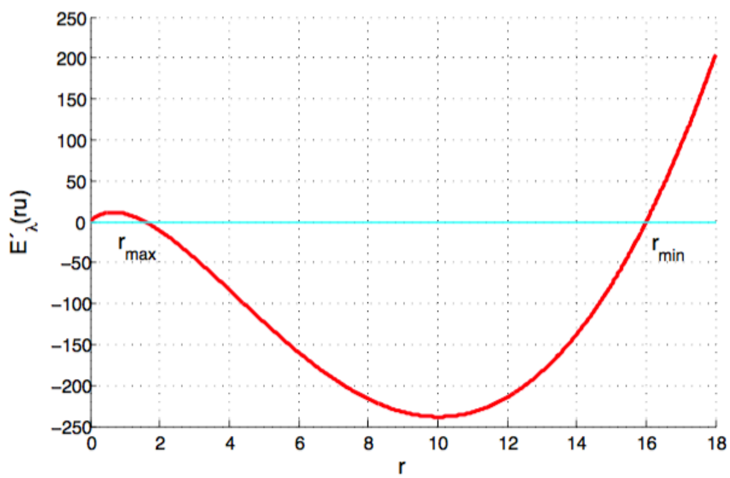
Concerning the stationary problem $SP(\alpha,\beta,\lambda)$ we recall that if $u \in \mathrm{H}^1_0(\Omega)\cap\mathrm{L}^\infty(\Omega)$ is a weak solution of $SP(\alpha,\beta,\lambda)$ then, by standard regularity results, $u \in \mathrm{W}^{2,p}(\Omega)$ for any $p \in (1,\infty)$ and then $u \in \mathrm{C}^{1,\gamma}(\overline{\Omega})\cap\mathrm{C}^2(\Omega)$ for some $\gamma \in (0,1)$.

Due to the homogeneity of the involved non linear terms, our stability analysis will use the auxiliary real functionals $E'_\lambda(u)$ and $E''_\lambda(u)$ defined, for a given $u \in \mathrm{H}^1_0(\Omega)$, by

$$E'_\lambda(u) := \frac{\partial}{\partial r}E_\lambda(ru)|_{r=1} \tag{8}$$

and

$$E''_\lambda(u) := \frac{\partial^2}{\partial r^2} E_\lambda(ru)|_{r=1}. \tag{9}$$

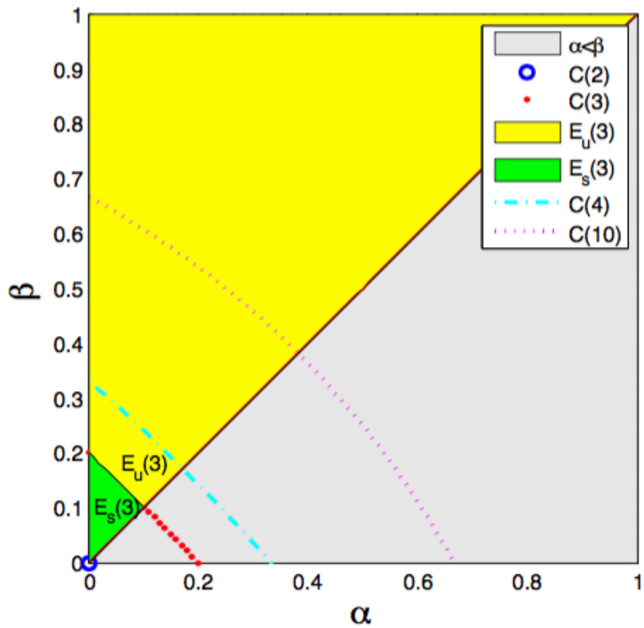


$r_{\min}$ and $r_{\max}$

Notice that any weak solution $u \in \mathrm{H}_0^1(\Omega)$ of $SP(\alpha, \beta, \lambda)$ satisfies that $E'_\lambda(u) = 0$ (as one gets trivially by taking u as a test function). We also point out that E'_λ and E''_λ should not be confused with the first and second variation of the functional DE_λ and D^2E_λ involving dual spaces. It is not difficult to show that in case $\beta < 1$ the real equation $E'_\lambda(ru) = 0$ may have at most two nonzero roots $r_{\min} > 0$ and $r_{\max} > 0$ such that $E''_\lambda(r_{\max}) \geq 0$, $E''_\lambda(r_{\min}) \leq 0$ and $0 < r_{\max} \leq r_{\min}$ (see Figure 2), whereas, in case $\beta = 1$ the equation $E'_\lambda(r) = 0$ has precisely one nonzero root $r_{\max} > 0$ such that $E''_\lambda(r_{\max}) \leq 0$, for any $\lambda > \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian with Dirichlet boundary conditions. This implies that any weak solution of $SP(\alpha, \beta, \lambda)$ (any critical value of $E_\lambda(u)$) corresponds to one of the cases $r_{\min} = 1$ or $r_{\max} = 1$. However, it was discovered in [44] (see also [42]) that in case when we study flat and compactly supported solutions this correspondence essentially depends on a relation between α, β and N.

In the present paper, which develops and improves [44], we introduce the following critical exponents curve depending on the dimension N

$$\mathcal{C}(\mathrm{N}) := \{(\alpha, \beta) \in \mathcal{E} : \ \ 2(1+\alpha)(1+\beta) - \mathrm{N}(1-\alpha)(1-\beta) = 0\} \tag{10}$$



Sets $\mathcal{E}_s(\mathrm{N})$ and $\mathcal{E}_u(\mathrm{N})$ for $\mathrm{N} = 3,\ 4$ and 10

in the set of relevant exponents $\mathcal{E} := \{(\alpha,\beta) :\ 0 < \alpha < \beta < 1\}$. This curve $\mathcal{C}(\mathrm{N})$ exists if and only if $\mathrm{N} \geq 3$, $\beta < 1$ and it separates two sets of exponents in $\mathcal{E}$ (see Figure 3)

$$\mathcal{E}_s(\mathrm{N}) := \{(\alpha,\beta) \in \mathcal{E} :\ \ 2(1+\alpha)(1+\beta) - \mathrm{N}(1-\alpha)(1-\beta) < 0\},$$

$$\mathcal{E}_u(\mathrm{N}) := \{(\alpha,\beta) \in \mathcal{E} :\ \ 2(1+\alpha)(1+\beta) - \mathrm{N}(1-\alpha)(1-\beta) > 0\},$$

whereas in the cases $\mathrm{N} = 1, 2$ or/and $\beta = 1$ one has $\mathcal{E} = \mathcal{E}_u(\mathrm{N})$.

The main property of $\mathcal{C}(\mathrm{N})$ is contained in the following result:

Lemma 1 *Let* $\mathrm{N} \geq 1$, $\lambda > 0$, $0 < \alpha < \beta < 1$ *and let* Ω *be a bounded and star-shaped domain in* $\mathbb{R}^{\mathrm{N}}$ *whose boundary* $\partial\Omega$ *is a* C^1*-manifold.*

(1^o) *Assume* $(\alpha,\beta) \in \mathcal{C}(\mathrm{N})$. *Then any flat or compactly supported solution* u *of* (2) *satisfies* $E''_\lambda(u) = 0$.

(2^o) *Assume* $(\alpha,\beta) \in \mathcal{E}_u(\mathrm{N})$. *Then any flat or compactly supported solution solution* u *of* (2) *satisfies* $E''_\lambda(u) < 0$.

(3^o) *Assume* $(\alpha,\beta) \in \mathcal{E}_s(\mathrm{N})$. *Then any solution* u *of* (2) *satisfies* $E''_\lambda(u) > 0$.

We recall that the existence of positive, flat or compactly supported ground state solutions of (2) in the case $\beta < 1$, $\mathrm{N} \geq 3$ and $(\alpha,\beta) \in \mathcal{E}_s(\mathrm{N})$ has been obtained in [44]. Furthermore, some results on the existence of flat and compactly supported solutions of (2) (not necessary ground states) in case $\mathrm{N} \geq 1$, $0 < \alpha < \beta \leq 1$ can be found in [26, 28, 46, 47] (see also [53]).

As already mentioned, one of the main goals of this paper is to study the H^1_0-stability of flat ground state solutions of $SP(\alpha,\beta,\lambda)$. We recall that, if $v(t;v_0)$ is a weak solution to $PP(\alpha,\beta,\lambda,v_0)$, we shall say that $v(t;v_0)$ is H^1_0*-stable* if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$||v(t;v_0) - v(t;w_0)||_1 < \varepsilon \ \text{ for any } w_0 \text{ such that } \ ||v_0 - w_0||_1 < \delta, \ \ \forall t > 0, \tag{11}$$

where we used the $\mathrm{H}^1_0(\Omega)-$norm

$$||u||_1 = \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2}.$$

Furthermore, we will use also the following definition: a solution u_λ of $SP(\alpha,\beta,\lambda)$ is said to be *linearly unstable solution* if $\lambda_1(-\Delta+\alpha u_\lambda^{\alpha-1}-\lambda\beta u_\lambda^{\beta-1}:\text{support}(u_\lambda))<0$. Given ω open subset of Ω we define $\lambda_1(-\Delta+\alpha u_\lambda^{\alpha-1}-\lambda\beta u_\lambda^{\beta-1}:\overline{\omega})$ as the first eigenvalue of the problem

$$\begin{cases} -\Delta\psi-(\lambda\beta u_\lambda^{\beta-1}-\alpha u_\lambda^{\alpha-1})\psi=\mu\psi & \text{in } \omega,\\ \psi=0 & \text{on } \partial\omega. \end{cases} \tag{12}$$

Notice that since u_λ may have compact support on Ω and $0<\alpha<\beta\leq 1$ we must take $\overline{\omega}=$support(u_λ) in order to give a sense to the linearized operator $-\Delta\psi-(\lambda\beta u_\lambda^{\beta-1}-\alpha u_\lambda^{\alpha-1})\psi$. Remember that in our case support(u_λ) is a ball if support$(u_\lambda)\subsetneq\overline{\Omega}$. We take $\omega=\Omega$ if u_λ is neither a flat nor any compactly supported solution of $SP(\alpha,\beta,\lambda)$. One can guess that some additional comparison argument would actually provide unstability in the meaning of Lyapounov (see Section 5 of [6] for this and related matters).

In what follows, we will also use the following definition ([6], [39]): a solution $v(t;v_0)$ of $PP(\alpha,\beta,\lambda,v_0)$ is said to be *globally* $\mathrm{H}_0^1(\Omega)$*-unstable* if for any $\delta>0$ there exists

$$w_0\in U_\delta(v_0):=\{w\in H_0^1(\Omega):\ ||v_0-w||_1<\delta\}$$

such that

$$||v(t;v_0)-v(t;w_0)||_1\to\infty \ \text{ as } \ t\to\infty. \tag{13}$$

We shall use also the following definition: a ground state u_λ of $SP(\alpha,\beta,\lambda)$ is said to be *isolated* if , there exists a neighborhood

$$U_\delta(u_\lambda):=\{v\in H_0^1(\Omega),\ v\geq 0 \text{ on } \Omega,\ \text{such that } ||u_\lambda-v||_1<\delta\},$$

with $\delta>0$ such that $SP(\alpha,\beta,\lambda)$ has no other ground state in $U_\delta(u_\lambda)\setminus u_\lambda$.

Moreover, given $\lambda>0$, it is useful to introduce the notation

$$\mathbb{G}_\lambda=\{u_\lambda \text{ is a ground state of } SP(\alpha,\beta,\lambda)\}.$$

It is not difficult to see that for any $\lambda>0$ the set $\mathbb{G}_\lambda$ is a bounded subset of $H_0^1(\Omega)$ but the question about for which values of λ the set $\mathbb{G}_\lambda$ is non-empty is quite delicate (see Remark 1). Since in some cases $\mathbb{G}_\lambda$ may contain a continuum of solutions it is convenient to extend the above notion of H_0^1-stability, which is well adapted only for isolated solutions of $SP(\alpha,\beta,\lambda)$, to a more general context. Given $\delta>0$ we first introduce the notation

$$V_\delta(\mathbb{G}_\lambda):=\{v\in H_0^1(\Omega):\inf_{u\in\mathbb{G}_\lambda}||u-v||_1<\delta\}.$$

Then we introduce here the following definition: a set $G_\lambda\subset\mathbb{G}_\lambda$ of ground states of $SP(\alpha,\beta,\lambda)$ is said to be H_0^1*-stable* if, given any $\varepsilon>0$, there exists $\delta>0$ such that

$$\inf_{v_0\in G_\lambda}||v_0-v(t;w_0)||_1<\varepsilon \ \text{ for any } w_0\ \in V_\delta(G_\lambda),\ \ \forall t>0. \tag{14}$$

Notice that if u_λ is an isolated solution of $SP(\alpha,\beta,\lambda)$ then taking $G_\lambda=\{u_\lambda\}$ the above notion of H_0^1-stability for G_λ coincides with the one given for a single solution u_λ.

Our first result concerns the existence of nonnegative ground states of (2) for the case $0<\alpha<\beta<1$. We have

Theorem 1 *Let* $\mathrm{N}\geq 1$, $0<\alpha<\beta<1$, Ω *be a bounded domain in* $\mathbb{R}^{\mathrm{N}}$*, with a smooth boundary. Then there exists* $\lambda^{gr}>0$ *such that for all* $\lambda>\lambda^{gr}$ *we have:*

(1^o) *problem* (2) *has a ground state* u_λ *which is nonnegative in* Ω *and* $u_\lambda\in\mathrm{C}^{1,\kappa}(\overline{\Omega})\cap\mathrm{C}^2(\Omega)$ *for some* $\kappa\in(0,1)$*;*

(2^o) *the set* $G_\lambda=\{u_\lambda\in\mathbb{G}_\lambda$ *such that* $E''_\lambda(u_\lambda)>0\}$ *is* $\mathrm{H}_0^1(\Omega)$*-stable. In particular, any isolated ground state* u_λ *such that* $E''_\lambda(u_\lambda)>0$ *is* $\mathrm{H}_0^1(\Omega)$*-stable;*

(3^o) *any nonnegative weak solution* u_λ *of* (2) *such that* $E''_\lambda(u_\lambda)<0$ *is a linearized unstable solution.*

As indicated before, in case $\beta = 1$ any nonnegative ground solution satisfies that $E''_\lambda(r_{\max}) \leq 0$ so, at least formally, all solutions are $\mathrm{H}_0^1(\Omega)$-unstable solutions. In fact, our main result for $\beta = 1$ proves the global $\mathrm{H}_0^1(\Omega)$-unstability for any ground solution. We also present here a sharper version of the existence of flat and compactly supported ground state solutions improving Theorem 5.1 of [28].

Theorem 2 *Let $\mathrm{N} \geq 1$, $\beta = 1$, $0 < \alpha < 1$, Ω be a bounded star-shaped domain in $\mathbb{R}^{\mathrm{N}}$, with a smooth boundary. Then for all $\lambda > \lambda^{gr} \equiv \lambda_1$ there hold*

(1^o) *problem (2) has a ground state u_λ which is nonnegative in Ω and $u \in \mathrm{C}^{1,\kappa}(\overline{\Omega}) \cap \mathrm{C}^2(\Omega)$ for some $\kappa \in (0,1)$. Moreover, there exists $\lambda^* > \lambda_1$ such that, problem (2) has a flat ground state u_{λ^*} if and only if Ω is a ball, and for the general case of Ω strictly star-shaped domain, for $\lambda > \lambda^*$ problem (2) has a compactly supported ground state solution;*

(2^o) *any ground state u_λ is globally $\mathrm{H}_0^1(\Omega)$-unstable.*

The study of the $\mathrm{H}_0^1(\Omega)$-stability of flat and compactly supported ground state solutions when $0 < \alpha < \beta < 1$ is more delicate by different reasons (see Remark 1). Notice that Theorem 2 applies to any kind of ground states, once $\beta = 1$, and so to ground states satisfying (3).

Theorem 3 *Let $\mathrm{N} \geq 1$, Ω be a bounded domain in $\mathbb{R}^{\mathrm{N}}$ whose boundary $\partial\Omega$ is a C^1-manifold.*

(I) *Assume $(\alpha, \beta) \in \mathcal{E}_u(\mathrm{N})$. Then any nonnegative flat or compactly supported solution u_λ of (2) is a linearized unstable solution.*

(II) *Assume $(\alpha, \beta) \in \mathcal{E}_s(\mathrm{N})$ and Ω is a strictly star-shaped domain with respect to the origin. Then there exists $\lambda^* > \lambda^{gr}$ such that*

(1^o) *for $\lambda > \lambda^*$ problem (2) has a compactly supported weak solution. Moreover, problem (2) has a flat ground state u_{λ^*} if and only if Ω is a ball;*

(2^o) *the set $G_\lambda = \{u_\lambda \in \mathbb{G}_\lambda$ and u_λ is flat or compactly supported solution of $SP(\alpha, \beta, \lambda)\}$ is $\mathrm{H}_0^1(\Omega)$-stable.*

Remark 1 *By Corollary 12 of [44], there exists $\Lambda_0 > \lambda^* > \lambda^{gr}$ such that if $\lambda \geq \Lambda_0$ and $(\alpha, \beta) \in \mathcal{E}_s(\mathrm{N})$ (i.e. if $D < 0$), then no ground state may be a flat neither a compactly supported weak solution. It is an open question to know if for $\lambda \in (\lambda^*, \Lambda_0)$ the compactly supported weak solutions are or not ground states. Nevertheless, the flat ground state u_{λ^*} may be rescaled originating a compactly supported solution for $\lambda > \lambda^*$(see Corollary 3 bellow). This shows that the rescaled solution is a "relatively-ground" state in the sense that its energy attains a minimum on the set of weak solutions with compact support. This could produce some partial stability result by adapting the proof of Theorem 3 (II) (2^o) to this peculiar class of stationary solutions but we shall not enter into details here.*

Finally, we point out that the limit case $\alpha = 0$ can be also considered. In particular, this shows that the first "compressed mode" function (solution of $SP(0, 1, \lambda)$, of great relevance in signal processing: see [48], [49]) is globally $\mathrm{H}_0^1(\Omega)$-unstable.

2 Parabolic problem. Existence, uniqueness and boundedness on non-negative solutions

Given $v_0 \in \mathrm{L}^\infty(\Omega)$, $v_0 \geq 0$, we shall say that $v \in \mathrm{C}([0, +\infty); \mathrm{L}^2(\Omega))$ is a weak solution of $PP(\alpha, \beta, \lambda, v_0)$ if $v \geq 0$, $\lambda v^\beta - v^\alpha \in \mathrm{L}^\infty((0, T) \times \Omega)$, for any $T > 0$ and

$$v(t) = T(t)v_0 + \int_0^t T(t-s)(\lambda v^\beta(s) - v^\alpha(s))ds. \tag{15}$$

Here $(T(t))_{t\geq 0}$ is the heat semigroup with homogeneous Dirichlet boundary conditions, i.e. $T(t) = e^{t(-\Delta)}$. The existence of weak solutions is an easy variation of previous results in the literature (see, e.g. [14], [3] and the works [20], [19] dealing with the more difficult case of singular equations $\alpha \in (-1, 0)$). For the reader convenience we shall collect here some additional regularity information on weak solutions of $PP(\alpha, \beta, \lambda, v_0)$.

Proposition 1 *For any $v_0 \in \mathrm{L}^\infty(\Omega)$, $v_0 \geq 0$ there exists a nonnegative weak solution $v \in \mathcal{C}([0, +\infty), \mathrm{L}^2(\Omega))$ of $PP(\alpha, \beta, \lambda, v_0)$. In fact, for every $p \in [1, \infty]$, $v \in \mathcal{C}([0, +\infty); \mathrm{L}^p(\Omega))$, and if $p < \infty$*

$$v - T(.)v_0 \in \mathrm{L}^p(\tau, T; \mathrm{W}^{2,p}(\Omega) \cap \mathrm{W}_0^{1,p}(\Omega)) \cap \mathrm{W}^{1,p}(\tau, T; \mathrm{L}^p(\Omega)), \tag{16}$$

for any $0 < \tau < T$. In particular, v satisfies the equation $PP(\alpha, \beta, \lambda, v_0)$ for a.e. $t \in (0, +\infty)$. Moreover, if we also assume that $v_0 \in \mathrm{H}_0^1(\Omega)$ then $\frac{\partial}{\partial t} E_\lambda(v(.)) \in \mathrm{L}^1(\tau, T)$, the function $E_\lambda(v(.))$ is absolutely continuous for a.e. $t \in (\tau, T)$ and

$$\frac{\partial}{\partial t} E_\lambda(v(t)) = -\int_\Omega v_t(t)^2 dx. \tag{17}$$

PROOF. Among many possible methods to prove the existence of weak solutions we shall follow here the one based on a fixed point argument as in [33] (see also [32] where the case $\beta = 0$ was considered on a Riemannian manifold). For every $h \in \mathrm{L}^\infty((0, T) \times \Omega)$ we consider the problem (P_h)

$$(P_h) \quad \begin{cases} v_t - \Delta v + |v|^{\alpha-1} v = h & \text{in } (0, +\infty) \times \Omega, \\ v = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ v(0, x) = v_0(x) & \text{on } \Omega, \end{cases}$$

which we can reformulate in terms of an abstract Cauchy problem on the Hilbert space $\mathrm{H} = \mathrm{L}^2(\Omega)$ as

$$(P_h) = \begin{cases} \dfrac{dv}{dt}(t) + \mathcal{A}v(t) = h(t) & t \in (0, T), \text{ in H}, \\ v(0) = v_0 \end{cases}$$

where $\mathcal{A} = \partial\varphi$ denotes the subdifferential of the convex function

$$\varphi(v) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega |\nabla v|^2 \, dx + \dfrac{1}{\alpha+1} \displaystyle\int_\Omega |v|^{\alpha+1} \, dx & \text{if } v \in \mathrm{H}_0^1(\Omega) \cap \mathrm{L}^{\alpha+1}(\Omega), \\ +\infty & \text{otherwise}, \end{cases}$$

(see, e.g. [8], [7] and [22]). As in [33], [32], we define the operator $\mathcal{T} : h \to g$ where $g = \lambda |v_h|^{\beta-1} v_h$ and v_h is the solution of (P_h). It is easy to see that every fixed point of $\mathcal{T}$ is a solution of $PP(\alpha, \beta, \lambda, v_0)$. Then $\mathcal{T}$ satisfies the hypotheses of Kakutani Fixed Point Theorem (see e.g. Vrabie [55]), since if $X = \mathrm{L}^2(0, T; \mathrm{L}^2(\Omega))$ then

(i) $\mathcal{K} = \{h \in \mathrm{L}^2(0, T; \mathrm{L}^\infty(\Omega)) : ||h(t)||_{\mathrm{L}^\infty(\Omega)} \leq C_0$ a.e. $t \in (0, T)\}$ is a nonempty, convex and weakly compact set of X;

(ii) $\mathcal{T} : \mathcal{K} \mapsto 2^X$ with nonempty, convex and closed values such that $\mathcal{T}(g) \subset \mathcal{K}, \forall g \in \mathcal{K}$;

(iii) graph($\mathcal{T}$) is weakly$\times$weakly sequentially closed.

Consequently, $\mathcal{T}$ has at least one fixed point in $\mathcal{K}$ which is a local (in time) solution of $PP(\alpha, \beta, \lambda, v_0)$. The final key point is to show that there is no blow-up phenomenon. This holds by the a priori estimate

$$0 \leq v(t, x) \leq z(t, x), \text{ for any } t \in [0, +\infty) \times \Omega,$$

where $v(t, x)$ is any weak solution of $PP(\alpha, \beta, \lambda, v_0)$ and $z(t, x)$ is the solution of the corresponding auxiliary problem

$$\begin{cases} z_t - \Delta z = \lambda z^\beta & \text{in } (0, +\infty) \times \Omega \\ z = 0 & \text{on } (0, +\infty) \times \partial\Omega \\ z(0, x) = v_0(x) & \text{on } \Omega. \end{cases} \tag{18}$$

This implies that there is no finite blow-up (and thus the maximal existence time is $T_{\max} = +\infty$). In particular, if $\beta \in (0,1)$ we have the estimate

$$\|v(t)\|_{\mathrm{L}^\infty(\Omega)} \leq \left(\|v_0\|^{1-\beta}_{\mathrm{L}^\infty(\Omega)} + (1-\beta)t \right)^{1/(1-\beta)}.$$

If $\beta = 1$ then the function $w(t,x) = v(t,x)e^{-\lambda t}$ satisfies

$$\left\{ \begin{array}{ll} w_t - \Delta w + e^{-\lambda(1-\alpha)t} w^\alpha = 0 & \text{in } (0,+\infty) \times \Omega \\ w = 0 & \text{on } (0,+\infty) \times \partial\Omega \\ w(0,x) = v_0(x) & \text{on } \Omega, \end{array} \right. \tag{19}$$

which is uniformly (pointwise) bounded by the solution of the linear heat equation with the same initial datum. Since the operator $A = \overline{\partial\varphi}^{\mathrm{L}^p(\Omega)\times\mathrm{L}^p(\Omega)}$ is m-accretive in $\mathrm{L}^p(\Omega)$ for every $p \in [1,\infty]$ (see, e.g. the presentation made in [22]), by the regularity results for semilinear accretive operators we conclude the first part of the additional regularity of the statement (16). Finally, by Theorem 3.6 of [8] we know that $\frac{\partial}{\partial t}\varphi(v_h) \in \mathrm{L}^1(\tau,T)$, $\varphi(v_h)$ is absolutely continuous and for a.e. $t \in (\tau,T)$

$$\frac{\partial}{\partial t}\varphi(v_h) = \int_\Omega (h(t))(v_h)_t(t)dx - \int_\Omega \left[(v_h)_t(t)\right]^2 dx.$$

Then (17) holds by taking $h = \lambda|v_h|^{\beta-1}v_h$ (the fixed point of $\mathcal{T}$). □

Proposition 2 *Assume $0 < \alpha < \beta \leq 1$. Then the weak solution is unique.*

PROOF. The argument is quite direct if $\beta = 1$. Indeed, thanks to the change of variable $w(t,x) = v(t,x)e^{-\lambda t}$ the problem becomes (19) and the result follows from the semigroup theory since it is well-known (see, e.g., [22] Chapter 4) that the operator $Aw := -\Delta w + e^{-\lambda(1-\alpha)t}|w|^{\alpha-1}w$ is a T-accretive operator in $\mathrm{L}^p(\Omega)$ for any $p \in [1,+\infty]$. For the case $0 < \alpha < \beta < 1$ we multiply by $(w(t) - v(t))_+$ the difference of the inequalities satisfied by two possible nonnegative weak solutions w and v. We obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega [w(t)-v(t)]_+^2 + \int_\Omega |\nabla[w(t)-v(t)]_+|^2 \\ &\leq \lambda \int_{\{w>v\}} (w(t)^\beta - v(t)^\beta)[w(t)-v(t)] - \int_\Omega (w(t)^\alpha - v(t)^\alpha)[w(t)-v(t)]_+. \end{aligned}$$

But, since $0 < \alpha < \beta < 1$ the function

$$f(v) := \lambda|v|^{\beta-1}v - |v|^{\alpha-1}v$$

is such that

$$\sup_{u\neq 0} f'(u) \leq M$$

for some $M > 0$. Then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [w(t)-v(t)]_+^2 + \int_\Omega |\nabla[w(t)-v(t)]_+|^2 \leq M\int_\Omega [w(t)-v(t)]_+^2$$

and the conclusion follows from Gronwall's inequality.

□

Remark 2 *Assume*

$$v_0(x) \geq K_0 d(x)^{2/(1-\alpha)} \text{ for any } x \in \overline{\Omega}, \tag{20}$$

for some constant $K_0 > 0$, then it is possible to construct a (local) subsolution showing that if v is the weak solution of $PP(\alpha,\beta,\lambda,v_0)$ we have:

a) given $T > 0$ for any $K_0 > 0$ there is a $T_0 = T_0(K_0) \in (0,T]$ such that $\forall t \in (0,T_0)$, $v(t,x) \geq C(T_0)d(x)^{\nu}$ in Ω with $\delta(x) := \mathrm{dist}(x, \partial\Omega)$ and $\nu = 2/(1-\alpha)$.
b) if K_0 and λ are large enough then the above estimate holds for any $T > 0$ (in the sense that $C(T_0)$ is independent of T_0).
In consequence, if $u \in \mathrm{L}^{\infty}(\Omega)$ is a solution of the stationary problem $SP(\alpha,\beta,\lambda)$ such that $v(t) \to u$ in $\mathrm{L}^2(\Omega)$ a.e. $t \nearrow +\infty$, then u satisfies the nondeneracy property $u(x) \geq Kd(x)^{2/(1-\alpha)}$ for some $K > 0$.
The proof is an adaptation of the techniques presented in [24], [25] (see also some related local subsolutions in [1] and [31]).

The stability of the trivial solution $u \equiv 0$ of $SP(\alpha,\beta,\lambda)$ for λ small is very well illustrated by means of the following "extinction in finite time" property of solutions of the associated parabolic problem $PP(\alpha,\beta,\lambda,v_0)$ assumed λ small enough.

Theorem 4 *Assume*

$$\lambda \in [0, \lambda_1). \tag{21}$$

Let $v_0 \in \mathrm{L}^{\infty}(\Omega)$, $v_0 \geq 0$. Then there exists $T_0 > 0$ such that the solution v of $PP(\alpha,\beta,\lambda,v_0)$ satisfies that $v(t) \equiv 0$ on Ω for any $t \geq T_0$.

PROOF. We shall use an energy method in the spirit of [2] (see also [34]). By multiplying by $v(t)$ and integrating by parts (as in the proof of uniqueness) we arrive to

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} v(t)^2 dx + \int_{\Omega} |\nabla v(t)|^2 \, dx + \int_{\Omega} v(t)^{\alpha+1} dx = \lambda \int_{\Omega} v(t)^{\beta+1} dx.$$

Assume now that $\beta = 1$. Then, by using the Poincaré inequality

$$\lambda_1 \int_{\Omega} v(t)^2 dx \leq \int_{\Omega} |\nabla v(t)|^2 \, dx \tag{22}$$

we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} v(t)^2 dx + \left(1 - \frac{\lambda}{\lambda_1}\right)\int_{\Omega} |\nabla v(t)|^2 \, dx + \int_{\Omega} v(t)^{\alpha+1} dx \leq 0$$

and the result holds exactly as in Proposition 1.1, Chapter 2 of [2]. Indeed, by applying the Gagliardo-Nirenberg inequality,

$$\left[\int_{\Omega} v^r dx\right]^{1/r} \leq C \left[\int_{\Omega} |\nabla v|^2 \, dx\right]^{\theta/2} \left[\int_{\Omega} v dx\right]$$

for any $r \in [1, +\infty)$ if $\mathrm{N} \leq 2$ and $r \in \left[1, \dfrac{2\mathrm{N}}{\mathrm{N}-2}\right]$ if $\mathrm{N} > 2$ (with $\theta = \dfrac{2\mathrm{N}(r-1)}{r+2\mathrm{N}} \in (0,1)$), we have that the function

$$y(t) := \frac{d}{dt}\int_{\Omega} v(t)^2 dx$$

satisfies the inequality

$$y'(t) + Cy^{\upsilon}(t) \leq 0$$

for some $C > 0$ and $\upsilon \in (0,1)$. If $\beta \in (0,1)$ then we introduce the change of unknown $v = \mu\widehat{v}$ getting

$$\mu\widehat{v}_t - \mu\Delta\widehat{v} + \mu^{\alpha}\widehat{v}^{\alpha} = \lambda\mu^{\beta}\widehat{v}^{\beta}.$$

By choosing μ such that

$$\mu < \frac{1}{\lambda_1^{\frac{1}{\beta-\alpha}}}$$

we can assume without loss of generality that $\lambda < \min(\lambda_1, 1)$. Moreover, since

$$\lambda \int_{\Omega} v(t)^{\beta+1} dx \leq \lambda \int_{\Omega} v(t)^2 dx + \lambda \int_{\Omega} v(t)^{\alpha+1} dx,$$

we get that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v(t)^2 dx + \left(1-\frac{\lambda}{\lambda_1}\right)\int_\Omega |\nabla v(t)|^2\, dx + (1-\lambda)\int_\Omega v(t)^{\alpha+1} dx \le 0,$$

and the proof ends as in the preceding case. □

Remark 3 *The assumption* (21) *is optimal if* $\beta = 1$*: indeed, by the results of [27] we know that for any* $\lambda > \lambda_1$ *there exists a non-negative nontrivial solution* u *of the associated stationary problem* $SP(\alpha, 1, \lambda)$*.*

In fact, for any $\lambda > 0$ the trivial solution $u \equiv 0$ of the stationary problem $SP(\alpha, \beta, \lambda)$ is asymptotically $\mathrm{L}^\infty(\Omega)$-stable in the sense that it attracts solutions of $PP(\alpha, \beta, \lambda, v_0)$, in $\mathrm{L}^\infty(\Omega)$, for small initial data v_0.

Proposition 3 *Let* $v_0 \in \mathrm{L}^\infty(\Omega)$, $v_0 \ge 0$*. Given* $\lambda > 0$ *assume that*

$$\|v_0\|_{\mathrm{L}^\infty(\Omega)} < \lambda^{-1/(\beta-\alpha)}.$$

Then $v(t) \to 0$ *in* $\mathrm{L}^\infty(\Omega)$ *as* $t \to +\infty$*.*

PROOF. Use the solution of the associated ODE (with $\|v_0\|_{\mathrm{L}^\infty(\Omega)}$ as initial datum) as supersolution. □

Concerning non-uniformly bounded trajectories we have:

Proposition 4 *Let* $v_0 \in L^\infty(\Omega)$, $v_0 \ge 0$ *such that*

$$0 < u_\lambda(x) + \varepsilon_0 \le v_0(x) \text{ a.e. } x \in \Omega, \tag{23}$$

for some $\varepsilon_0 > 0$ *and* u_λ *solution of the associated stationary problem* $SP(\alpha, \beta, \lambda)$ *such that*

$$meas\{x \in \Omega : u_\lambda(x) + \varepsilon_0 > \lambda^{-1/(\beta-\alpha)}\} > 0.$$

Assume $\beta = 1$ *or* (20)*. Then* $\|v(t)\|_{\mathrm{L}^\infty(\Omega)} \nearrow +\infty$ *as* $t \to +\infty$*.*

PROOF. Since obviously u_λ is a solution of $PP(\alpha, \beta, \lambda, u_\lambda)$ then we first get, by Theorem 2.1, that $u_\lambda(x) \le v(t, x)$ for any $t \in [0, +\infty)$ and a.e. $x \in \Omega$. Moreover, $u_\lambda(x) > \lambda^{-1/(\beta-\alpha)} > 0$ on a positively measured subset Ω_λ of Ω where we can apply the strong maximum principle to conclude that $u_\lambda(x) < v(t, x)$ for any $t \in [0, +\infty)$ and a.e. $x \in \Omega_\lambda$. Since $u_\lambda \in \mathrm{C}(\overline{\Omega})$ there exists $x_\lambda \in \overline{\Omega}_\lambda$ such

$$u_\lambda(x_\lambda) = \min_{\overline{\Omega}_\lambda} u_\lambda.$$

Taking now $U(t)$ as the solution of the ODE

$$ODE(\alpha, \beta, \lambda, u_\lambda(x_\lambda) + \varepsilon_0) \quad \begin{cases} U_t + U^\alpha = \lambda U^\beta & \text{in } (0, +\infty), \\ U(0) = u_\lambda(x_\lambda) + \varepsilon_0, \end{cases} \tag{24}$$

by the standard comparison principle (notice that now the involved nonlinearities are Lipschitz continuous on this set of values) we get that for any $t \in [0, +\infty)$

$$U(t) \le v(t, x) \ \text{ a.e. } x \in \Omega_\lambda.$$

Finally, since we know that $U(t) \nearrow +\infty$ as $t \to +\infty$, we get the result. □

3 Proof of Lemma 1: critical exponents curve on the plane $\alpha \times \beta$

In this section, using Pohozaev's identity [51] (see also [52]) and developing the spectral analysis by the fibering method [40], [41] (see also [43]) we introduce the critical exponents curve $\mathcal{C}(\mathrm{N})$ on the plane (α, β) and study its main properties.

From now on we will use the notations

$$T(u) = \int_\Omega |\nabla u|^2\, dx,\ A(u) = \int_\Omega |u|^{\alpha+1}\, dx,\ B(u) = \int_\Omega |u|^{\beta+1}\, dx.$$

Then

$$E_\lambda(u) = \frac{1}{2}T(u) + \frac{1}{\alpha+1}A(u) - \lambda\frac{1}{\beta+1}B(u).$$

Moreover, after defining $E'_\lambda(u) := \frac{\partial}{\partial r}E_\lambda(ru)|_{r=1}$ (8) we get that if u is a weak solution of $SP(\alpha,\beta,\lambda)$ then

$$E'_\lambda(u) = T(u) + A(u) - \lambda B(u) = 0.$$

We also point out that defining $E''_\lambda(u) = \frac{\partial^2}{\partial r^2}E_\lambda(ru)|_{r=1}$ then it is a routine matter to check that for any given $u \in \mathrm{H}^1_0(\Omega)\setminus\{0\}$

$$E''_\lambda(u) = T(u) + \alpha A(u) - \lambda\beta B(u).$$

3.1 Case $0 < \alpha < \beta < 1$

For any given $u \in \mathrm{H}^1_0(\Omega)\setminus\{0\}$ the equation

$$E'_\lambda(ru) = 0 \tag{25}$$

may have at most two roots $r_{max}(u), r_{\min}(u) \in \mathbb{R}^+$ such that $r_{max}(u) \le r_{\min}(u)$. Furthermore $r_{max}(u) < r_{\min}(u)$, if and only if

$$E''_\lambda(r_{max}(u)u) < 0 \ \text{ and } \ E''_\lambda(r_{\min}(u)u) > 0,$$

and $r_{max}(u) = r_{\min}(u) =: r_s(u)$ if and only if $E''_\lambda(r_s(u)\cdot u) = 0$ (see Figure 2). The following characteristic value was introduced in [44]:

$$\Lambda_0 = \inf_{u\in H^1_0(\Omega)\setminus 0} \lambda_0(u), \tag{26}$$

with

$$\lambda_0(u) = c_0^{\alpha,\beta}\lambda(u),$$

$$c_0^{\alpha,\beta} = \frac{(1-\alpha)(1+\beta)}{(1-\beta)(1+\alpha)}\left(\frac{(1+\alpha)(1-\beta)}{2(\beta-\alpha)}\right)^{\frac{\beta-\alpha}{1-\alpha}}$$

and

$$\lambda(u) = \frac{A(u)^{\frac{1-\beta}{1-\alpha}}T(u)^{\frac{\beta-\alpha}{1-\alpha}}}{B(u)}. \tag{27}$$

Note that by the Gagliardo-Nirenberg inequality (see [44, Proposition 2]) it follows that $0 < \Lambda_0 < +\infty$. In [44], it was proved the

Proposition 5 *If $\lambda \ge \Lambda_0$, then there exists $u \in H^1_0(\Omega)\setminus\{0\}$ such that $E'_\lambda(u) = 0$ and $E_\lambda(u) \le 0$, $E''_\lambda(u) > 0$.*

We need also the following characteristic value from [44]

$$\Lambda_1 = \inf_{u\in \mathrm{H}^1_0\setminus\{0\}} \lambda_1(u). \tag{28}$$

where

$$\lambda_1(u) = c_1^{\alpha,\beta}\lambda(u), \tag{29}$$

where

$$c_1^{\alpha,\beta} = \frac{1-\alpha}{1-\beta}\left(\frac{1-\beta}{\beta-\alpha}\right)^{\frac{\beta-\alpha}{1-\alpha}}. \tag{30}$$

As before we have $0 < \Lambda_1 < +\infty$. Furthermore, $0 < \Lambda_1 < \Lambda_0 < +\infty$ (see [44, Claim 2]) and we have as in Proposition 5 (see also [44])

Proposition 6 *If $\lambda > \Lambda_1$, then there exists $u \in \mathrm{H}_0^1(\Omega) \setminus \{0\}$ such that $E'_\lambda(u) = 0$, whereas if $\lambda < \Lambda_1$, then $E'_\lambda(u) > 0$ for any $u \in \mathrm{H}_0^1(\Omega) \setminus \{0\}$.*

Let $u \in \mathrm{H}_0^1(\Omega)$ be a weak solution of (2). Standard regularity arguments show that $u \in \mathrm{C}^{1,\gamma}(\overline{\Omega}) \cap \mathrm{C}^2(\Omega)$ for some $\gamma \in (0,1)$. Since by assumption $\partial\Omega$ is a C^1-manifold the Pohozaev's identity holds [51, 45], namely

$$P_\lambda(u) + \frac{1}{2\mathrm{N}}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 x\cdot\nu\, ds = 0, \tag{31}$$

where

$$P_\lambda(u) := \frac{\mathrm{N}-2}{2\mathrm{N}}T(u) + \frac{1}{\alpha+1}A(u) - \lambda\frac{1}{\beta+1}B(u), \quad u \in \mathrm{H}_0^1(\Omega). \tag{32}$$

Note that if Ω is a star-shaped (strictly star-shaped) domain with respect to the origin of $\mathbb{R}^\mathrm{N}$, then $x\cdot\nu \geq 0$ ($x\cdot\nu > 0$) for all $x \in \partial\Omega$. Thus we have

Proposition 7 *Assume that Ω is a star-shaped domain with respect to the origin of $\mathbb{R}^\mathrm{N}$, then $P_\lambda(u) \leq 0$ ($P_\lambda(u) = 0$) for any weak (flat or compactly supported) solution u of (2). If, in addition, Ω is strictly star-shaped, then a weak solution u of (2) is flat or it has compact support if and only if $P_\lambda(u) = 0$.*

Let us study now the critical exponent curve $\mathcal{C}(\mathrm{N})$ (see (10)) and prove Lemma 1. Consider the system (see [44])

$$\begin{cases} E'_\lambda(u) = \; T(u) + A(u) - \lambda B(u) = 0 \\ P_\lambda(u) = \dfrac{\mathrm{N}-2}{2\mathrm{N}}T(u) + \dfrac{1}{\alpha+1}A(u) - \lambda\dfrac{1}{\beta+1}B(u) = 0 \\ E''_\lambda(u) = T(u) + \alpha A(u) - \lambda\beta B(u) = 0. \end{cases} \tag{33}$$

This system is solvable with respect to the variables $T(u)$, $A(u)$, $B(u)$ if the corresponding determinant is non-zero. Defining

$$D = \frac{(\beta-\alpha)(2(1+\alpha)(1+\beta) - \mathrm{N}(1-\alpha)(1-\beta))}{2\mathrm{N}(1+\alpha)(1+\beta)} \tag{34}$$

such determinant is given by $-\lambda D$. On the other hand $D = 0$ if and only if $(\alpha,\beta) \in \mathcal{C}(\mathrm{N})$.

PROOF OF LEMMA 1. Let Ω be a star-shaped domain with respect to the origin of $\mathbb{R}^\mathrm{N}$. Then by Proposition 7 we have $P_\lambda(u) = 0$ for any flat or compactly supported solution u of (2). Note also that $E'_\lambda(u) = 0$. Thus, in case $(\alpha,\beta) \in \mathcal{C}(\mathrm{N})$, i.e. when the determinant of system (33) is equal to zero one has $E''_\lambda(u) = 0$ and we get the proof of statement (1^o) of Lemma 1. Observe that

$$\begin{aligned} D \cdot \tfrac{2\mathrm{N}(1+\alpha)}{(1-\alpha)[-2(1+\alpha)-\mathrm{N}(1-\alpha)]}B(u) = \quad & \frac{1}{1-\alpha}(E''_\lambda(u) - E'_\lambda(u)) - \\ & \frac{2N(1+\alpha)}{(\mathrm{N}-2)(1+\alpha)-2\mathrm{N}}(P_\lambda(u) - \frac{N-2}{2N}E'_\lambda(u)). \end{aligned}$$

Thus if $(\alpha,\beta) \in \mathcal{E}_u(\mathrm{N})$ and $P_\lambda(u) = 0$, $E'_\lambda(u) = 0$, then

$$E''_\lambda(u) = -D \cdot \frac{2\mathrm{N}(1+\alpha)}{2(1+\alpha)+\mathrm{N}(1-\alpha)}B(u) < 0$$

since $D > 0$ for $(\alpha,\beta) \in \mathcal{E}_u(\mathrm{N})$ and we obtain the proof of statement (2^o) of Lemma 1.

Under assumption (3^o) of Lemma 1, for a weak solution u of (2) we have $P_\lambda(u) \le 0$ (see Proposition 7) and therefore we get

$$E''_\lambda(u) \ge -D \cdot \frac{2\mathrm{N}(1+\alpha)}{2(1+\alpha)+\mathrm{N}(1-\alpha)} B(u) > 0,$$

since $D < 0$ for $(\alpha, \beta) \in \mathcal{E}_s(\mathrm{N})$. This completes the proof of Lemma 1. □

3.2 Case $\beta = 1$

Recall some results from [28]. In what follows (λ_1, φ_1) denotes the first eigenpair of the operator $-\Delta$ in Ω with zero boundary conditions. Let $u \in \mathrm{H}^1_0(\Omega)$. In this case we have

$$E_\lambda(ru) = \frac{r^2}{2}\mathrm{R}_\lambda(u) + \frac{r^{1+\alpha}}{1+\alpha}A(u)$$

where we denote

$$\mathrm{R}_\lambda(u) := \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \lambda \int_\Omega |u|^2 \,\mathrm{d}x.$$

Then

$$E'_\lambda(ru) = r\mathrm{R}_\lambda(u) + r^\alpha A(u)$$

and the equation $E'_\lambda(ru)$ has a positive solution only if both terms in $E'_\lambda(ru)$ have opposite sign, that is if and only if $\mathrm{R}_\lambda(u) < 0$. Note that there is $u \in \mathrm{H}^1_0(\Omega)$ such that $\mathrm{R}_\lambda(u) < 0$ if and only if $\lambda > \lambda_1$. It turns out that the only point $r(u)$ where $E'_\lambda(ru) = 0$ is given by

$$r(u) = \left(\frac{A(u)}{-\mathrm{R}_\lambda(u)}\right)^{1/(1-\alpha)}. \tag{35}$$

Furthermore, $E''_\lambda(r(u)u) < 0$ and

$$E_\lambda(r(u)u) = \max_{r>0} E_\lambda(ru). \tag{36}$$

Substituting (35) into $E_\lambda(ru)$ we obtain

$$J_\lambda(u) := E_\lambda(r(u)u) = \frac{(1-\alpha)}{2(1+\alpha)} \frac{A(u)^{\frac{2}{1-\alpha}}}{(-\mathrm{R}_\lambda(u))^{\frac{1+\alpha}{1-\alpha}}}. \tag{37}$$

Consider

$$\widehat{E}_\lambda = \min\{J_\lambda(u) : \ u \in \mathrm{H}^1_0(\Omega) \setminus \{0\},\ \mathrm{R}_\lambda(u) < 0\}. \tag{38}$$

It follows directly

Proposition 8 *A point $u \in \mathrm{H}^1_0(\Omega)$ is a minimizer of (38) if and only if $\tilde{u} = r(u)u$ is a ground state of (43).*

Remark 4 *We point out that in both cases, $\beta < 1$ and $\beta = 1$, the above results can be extended to the case in which the ground solution of $SP(\alpha, \beta, \lambda)$ minimizes the energy on the closed convex cone*

$$K = \{v \in \mathrm{H}^1_0(\Omega), v \ge 0 \quad on \quad \Omega\}. \tag{39}$$

Indeed, we introduce the modified energy functional

$$E^+_\lambda(u) = E_\lambda(u) + \int_\Omega j(u)dx$$

where

$$j(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that $j(ru) = j(u)$ for any $r > 0$. Obviously $E^+_\lambda(u) = E_\lambda(u)$ if $u \in K$. Moreover the additional term arising in the associated Euler-Lagrange equation, given by the subdifferential of the convex function $\int_\Omega j(u)dx$, vanishes when the ground state solution of $SP(\alpha, \beta, \lambda)$ is nonnegative.

4 Existence of ground state, flat and compactly supported ground state in the case $\beta < 1$

In this Section, we prove Theorem 1, (1^o) and Theorem 3, **(II)**, (1^o).

PROOF OF THEOREM 1, (1^o) Assume $\beta < 1$. In this case, the existence of a ground state of (2) when $(\alpha, \beta) \in \mathcal{E}_s(\mathrm{N})$ has been proved in [44]. The proof for the points $(\alpha, \beta) \in \mathcal{E} \setminus \mathcal{E}_s(\mathrm{N})$ can be obtained in a similar way. However for the sake of completeness, we present a summary of the proof.

Consider the constrained minimization problem of $E_\lambda(u)$ on the associated Nehari manifold

$$\hat{E}_\lambda := \min\{E_\lambda(u) : \ u \in \mathcal{N}_\lambda\} \tag{40}$$

where the admissible set of (40)

$$\mathcal{N}_\lambda := \{u \in H^1_0(\Omega) : \ E'_\lambda(u) = 0\}$$

is the corresponding Nehari manifold. Note that by Proposition 6, $\mathcal{N}_\lambda \neq \emptyset$ for any $\lambda > \Lambda_1$. Furthermore, by Sobolev's inequalities we have

$$E_\lambda(u) \geq \frac{1}{2}||u||_1^2 - c_1||u||_1^{1+\beta} \to \infty$$

as $||u||_1 \to \infty$. Thus $E_\lambda(u)$ is a coercive functional on $\mathrm{H}^1_0(\Omega)$. Using this it is not hard to prove the following (see also [44, Lemma 9])

Proposition 9 *Let $(\alpha, \beta) \in \mathcal{E}$. Then for any $\lambda \geq \Lambda_1$ problem (40) has a minimizer $u_\lambda \in \mathrm{H}^1_0(\Omega) \setminus \{0\}$, i.e. $E_\lambda(u_\lambda) = \hat{E}_\lambda$ and $u_\lambda \in \mathcal{N}_\lambda$.*

Let $\lambda \geq \Lambda_1$ and $u_\lambda \in \mathrm{H}^1_0(\Omega) \setminus \{0\}$ be a minimizer of (40). Then by the Lagrange multipliers rule there exist μ_1, μ_2 such that

$$\mu_1 DE_\lambda(u_\lambda) = \mu_2 DE'_\lambda(u_\lambda), \tag{41}$$

and $|\mu_1| + |\mu_2| \neq 0$. Thus, if $\mu_2 = 0$, then u_λ is a weak solution of (2).

This condition is satisfied under the assumptions of the following result:

Proposition 10 *Let $(\alpha, \beta) \in \mathcal{E}$. Then for any $\lambda \geq \Lambda_0$ (2) has a ground state u_λ which is nonnegative, $u \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for some $\gamma \in (0, 1)$ and $E''_\lambda(u_\lambda) > 0$.*

PROOF. Since $0 < \Lambda_1 < \Lambda_0$, then by Proposition 9 for any $\lambda \geq \Lambda_0$ there exists a minimizer $u_\lambda \in \mathrm{H}^1_0(\Omega) \setminus \{0\}$ of (40). Lemma 5 implies that there is $u \in \mathcal{N}_\lambda$ such that $E_\lambda(u) \leq 0$ and therefore $E_\lambda(u_\lambda) \leq E_\lambda(u) \leq 0$. This implies that $E''_\lambda(u_\lambda) > 0$. Let us test (41) by u_λ. Then

$$\mu_1 E'_\lambda(u_\lambda) = \mu_2(E''_\lambda(u_\lambda) + E'_\lambda(u_\lambda)).$$

Since $E'_\lambda(u_\lambda) = 0$, this yields that $\mu_2 E''_\lambda(u_\lambda) = 0$. But $E''_\lambda(u_\lambda) \neq 0$ and therefore $\mu_2 = 0$. Thus, by (41) we obtain $DE_\lambda(u_\lambda) = 0$, i.e u_λ is a weak solution of (2). Since any weak solution w_λ of (2) belongs to $\mathcal{N}_\lambda$, then (40) yields that u_λ is a ground state. The rest of the proposition is proved by standard arguments. □

From this Proposition, arguing by contradiction, it is not hard to show that there is an interval $(\Lambda_0 - \varepsilon, +\infty)$ for some $\varepsilon > 0$ such that for any $\lambda \in (\Lambda_0 - \varepsilon, +\infty)$ the minimizer u_λ of (40) satisfies $E''_\lambda(u_\lambda) > 0$. From this, as in the proof of Proposition 10, it follows that u_λ is a ground state of (2) which is nonnegative and $u \in \mathrm{C}^{1,\gamma}(\overline{\Omega}) \cap \mathrm{C}^2(\Omega)$ for some $\gamma \in (0, 1)$.

Thus we have a proof that there exists $\lambda^{gr} \in (\Lambda_1, \Lambda_0)$ such that for all $\lambda > \lambda^{gr}$ problem (2) has a ground state u_λ, which is nonnegative in Ω, $u \in \mathrm{C}^{1,\gamma}(\overline{\Omega}) \cap \mathrm{C}^2(\Omega)$ for some $\gamma \in (0, 1)$ and $E''_\lambda(u_\lambda) > 0$ This completes the proof of statement (1^o) of Theorem 1.

PROOF OF THEOREM 3, **(II)**, (1^o) By Corollary 15 from [44] it follows that there exists $\lambda^* > 0$ such that (2) has a compactly supported ground state u_{λ^*} which $u_{\lambda^*} \geq 0$ and $u_{\lambda^*} \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for some $\gamma \in (0, 1)$.

Coming back to the symmetry results mentioned in the Introduction we recall the following result which can be found in [53]: Consider the following auxiliary problem on the whole space $\mathbb{R}^{\mathrm{N}}$:

$$\begin{cases} -\Delta u + u^{\alpha} = u^{\beta} \ \text{ in } \ \mathbb{R}^{\mathrm{N}}, \\ |u| \to 0 \ \text{ as } \ |x| \to \infty, \\ u \geq 0 \ \text{ on } \ \mathbb{R}^{\mathrm{N}}. \end{cases} \tag{42}$$

Lemma 2 *Assume $0 < \alpha < \beta \leq 1$. Then any C^1 weak solution u of (42) has a compact support. Furthermore if we define*

$$\Theta := \{x \in \mathbb{R}^{\mathrm{N}} : u(x) > 0\}.$$

Then for every connected component Ξ of Θ we have

1. *Ξ is a ball;*
2. *u is radially symmetric with respect to the centre of the ball Ξ.*

This lemma immediately implies that (2) may has a flat ground state if and only if Ω is a ball.

5 Proof of Theorem 2, (1^o): existence of ground states and flat ground states in case $\beta = 1$

PROOF OF THEOREM 2, (1^o) The existence of a ground state for $\lambda > \lambda_1$ is obtained from the constrained minimization problem (38) and then using Proposition 8. The details of this proof was presented in [28, Theorem 2.1, p.6]. In order to prove the existence of a flat ground state we introduce the following auxiliary problem on the whole space $\mathbb{R}^{\mathrm{N}}$:

$$\begin{cases} -\Delta u + u^{\alpha} = u \ \text{ in } \ \mathbb{R}^{\mathrm{N}}, \\ |u| \to 0 \ \text{ as } \ |x| \to \infty, \\ u \geq 0 \ \text{ on } \ \mathbb{R}^{\mathrm{N}}. \end{cases} \tag{43}$$

Here and subsequently, $\mathrm{H}^1(\mathbb{R}^{\mathrm{N}})$ denotes the standard Sobolev space with the norm

$$||u||_1 = \left(\int_{\mathbb{R}^{\mathrm{N}}} |u|^2\,dx + \int_{\mathbb{R}^{\mathrm{N}}} |\nabla u|^2\,dx\right)^{1/2}.$$

Problem (43) has a variational form with the Euler-Lagrange functional

$$E(u) = \frac{1}{2}\mathrm{R}(u) + \frac{1}{\alpha+1}A(u), \quad u \in \mathrm{H}^1(\mathbb{R}^{\mathrm{N}})$$

where

$$\mathrm{R}(u) = \int_{\mathbb{R}^{\mathrm{N}}} |\nabla u|^2\,dx - \int_{\mathbb{R}^{\mathrm{N}}} |u|^2\,dx,\ A(u) = \int_{\mathbb{R}^{\mathrm{N}}} |u|^{\alpha+1}\,dx.$$

As above we call a nonzero weak solution u of (43) a ground state if it holds

$$E(u) \leq E(w)$$

for any nonzero weak solution w_λ of (43). Consider

$$E(ru) = \frac{r^2}{2}\mathrm{R}(u) + \frac{r^{1+\alpha}}{\alpha+1}A(u), \quad u \in \mathrm{H}^1(\mathbb{R}^{\mathrm{N}}), \ \ r \in \mathbb{R}^+.$$

For a given $u \in \mathrm{H}^1(\mathbb{R}^{\mathrm{N}})$ the equation

$$E'(ru) = r\mathrm{R}(u) + r^{\alpha}A(u) = 0, \quad r \in \mathbb{R}^+.$$

has only one root

$$r(u) = \left(\frac{A(u)}{-\mathrm{R}(u)} \right)^{1/(1-\alpha)} \tag{44}$$

which exists if and only if $\mathrm{R}(u) < 0$.
As above, substituting this root into $E_\lambda(ru)$ we obtain a zero-homogeneous functional

$$J(u) := E(r(u)u) = \frac{(1-\alpha)}{2(1+\alpha)} \frac{A(u)^{\frac{2}{1-\alpha}}}{(-\mathrm{R}(u))^{\frac{1+\alpha}{1-\alpha}}}. \tag{45}$$

We consider the problem

$$\min\{J(u): \ \ u \in \mathrm{H}^1(\mathbb{R}^\mathrm{N}) \setminus 0, \ \mathrm{R}(u) < 0\}, \tag{46}$$

and denote

$$\widehat{E}^\infty := \min\{J(u): \ \ u \in \mathrm{H}^1(\mathbb{R}^\mathrm{N}) \setminus 0, \ \mathrm{R}(u) < 0\}.$$

As in the precedent lines, we can prove

Proposition 11 *We have that u is a minimizer of (46) if and only if $\tilde{u} = r(u)u$ is a ground state of (43).*

In the Appendix below, using (46) we shall prove the following:

Lemma 3 *Assume $0 < \alpha < 1$. Then problem (43) has a classical nonnegative solution $u \in \mathrm{H}^1(\mathbb{R}^\mathrm{N})$ which is a ground state.*

Lemmas 2 and 3 yield

Corollary 1 *Assume $\beta = 1$, $0 < \alpha < 1$. Then there is a radius $R^* > 0$ such that problem (43) has a ground state u^* which is a flat classical radial solution and*

$$\mathrm{supp}(u^*) = B_{R^*}.$$

Let us return to problem (2). From Corollary 1 we have

Corollary 2 *Assume that $\beta = 1$ and $B_{R^*} \subset \Omega$. Then the ground state u_λ of (2) with $\lambda = 1$ coincides with the ground state u^* of (43) that is $u_\lambda|_{\lambda=1}$ is a compact support classical radial solution and*

$$\mathrm{supp}(u_\lambda)|_{\lambda=1} \equiv \bar{\Theta} = B_{R^*}.$$

PROOF. Any function w from $\mathrm{H}^1_0(\Omega)$ can be extended to $\mathbb{R}^\mathrm{N}$ as

$$\begin{cases} \tilde{w} = w & \text{in } \Omega, \\ \tilde{w} = 0 & \text{in } \mathbb{R}^\mathrm{N} \setminus \Omega, \end{cases} \tag{47}$$

Then $\tilde{w} \in \mathrm{H}^1(\mathbb{R}^\mathrm{N})$ and in this sense we may assume that $\mathrm{H}^1_0(\Omega) \subset \mathrm{H}^1(\mathbb{R}^\mathrm{N})$. Therefore by (38) we have

$$\widehat{E}^\infty \le \widehat{E}_1 \equiv \min\{J_1(v): \ \ v \in \mathrm{H}^1_0(\Omega) \setminus 0, \ \mathrm{H}_1(v) < 0\}.$$

Note that $u^* \in K \subset \mathrm{H}^1_0(B_{R^*}) \subset \mathrm{H}^1_0(\Omega)$ (remember that K was defined by (39)). This yields $\widehat{E}^\infty = E(u^*) = \widehat{E}_1$ and we get the proof. □

Assume now that Ω is a is star-shaped domain in $\mathbb{R}^\mathrm{N}$, with respect to the some point $z \in \mathbb{R}^\mathrm{N}$ which without loss of generality we may assume coincides with the origin $0 \in \mathbb{R}^\mathrm{N}$. Let u_λ be a ground state of (2) with $\beta = 1$. By making a change of variable $v_{\lambda(\kappa)}(y) = \kappa^{-2/(1-\alpha)} u_\lambda(\kappa y), \ \ y \in \Omega_\kappa$, as in [28], with $\kappa > 0$ we get

$$\begin{cases} -\Delta v_{\lambda(\kappa)} = \lambda(\kappa) v_{\lambda(\kappa)} - v^\alpha_{\lambda(\kappa)} & \text{in } \Omega_\kappa, \\ v_{\lambda(\kappa)} = 0 & \text{on } \Omega_\kappa. \end{cases} \tag{48}$$

where $\lambda(\kappa) = \lambda \kappa^2$, $\Omega_\kappa = \{y \in \mathbb{R}^N: \ y = x/\kappa, \ x \in \Omega\}$. Since u_λ is a ground state of (2), then it is easy to see that $v_{\lambda(\kappa)}$ is also a ground state of (48). Note that if $\kappa = \sqrt{1/\lambda}$ then $\lambda(\kappa) = 1$. On the other hand, if κ is sufficiently small then $B_{R^*} \subset \Omega_\kappa$. Hence by Corollary 1 there is a sufficiently large λ^* such that for any $\lambda > \lambda^*$ the ground state $v_{\lambda(\kappa)}$ with $\lambda(\kappa) = \lambda \cdot \kappa^2 = 1$ is compactly supported classical radial solution of (48) which coincides with the ground state u^* of (43). Thus we have proved

Corollary 3 *Assume $0 < \alpha < 1$. Then there exists $\lambda^* > 0$ such that for any $\lambda \geq \lambda^*$ problem (2) has a ground state u_λ which is a compactly supported classical radial solution. Furthermore, $u_\lambda(x) = \kappa^{2/(1-\alpha)}u^*(x/\kappa)$ where $\kappa = \sqrt{1/\lambda}$ with u^* a flat classical radial ground state of (43) on a ball.*

This completes the proof of Theorem 2, (1^o).

Remark 5 *Note that by [28, Lemma 3.3]*

$$\lambda^* > \lambda^c = (1 + \frac{2(1+\alpha)}{\mathrm{N}(1-\alpha)}) \cdot \lambda_1.$$

Furthermore, for any $\lambda \in (\lambda_1, \lambda^c)$ problem (2) cannot have any flat solution in $C^1(\overline{\Omega})$. At the same time, by Corollary 3 for any $\lambda > \lambda^*$ there exists $\sigma > 0$ such that for any $|y| < \sigma$ the function $u^y_\lambda := \kappa^{2/(1-\alpha)}u^*((x+y)/\kappa)$ is also a compactly supported classical ground state of (2). In other words, for $\lambda > \lambda^*$ problem (2) has a continuum set of compactly supported ground states.

6 Proof of Theorem 1, (2^o) and Theorem 3, (II), (2^o): Lyapunov stability

In this Section, first we prove statements of Theorem 1 (2^o) and then prove Theorem 3, **(II)**, (2^o). □

PROOF OF THEOREM 1, (2^o) *First step.* Assume $(\alpha, \beta) \in \mathcal{E}_s(\mathrm{N})$ and Ω is a strictly star-shaped domain with respect to the origin. We first prove that for any $\lambda > 0$ the set $\mathbb{G}_\lambda$ is bounded in H^1_0. Indeed, suppose by contrary that $||u_m||_1 \to \infty$ for some $(u_m) \subset \mathbb{G}_\lambda$. This is clearly a contradiction since $E_\lambda(u)$ is a coercive functional on the Nehari manifold.

Let

$$\widehat{G}_\lambda = \{u_\lambda \in \mathbb{G}_\lambda \text{ such that } E''_\lambda(u_\lambda) > 0\}.$$

Notice that since $(\alpha, \beta) \in \mathcal{E}_s(\mathrm{N})$ then $\widehat{G}_\lambda = \mathbb{G}_\lambda$. Let us show that E_λ is a Lyapunov function in the neighborhood $V_\delta(\widehat{G}_\lambda)$ if $0 < \delta < \delta_0$. We shall need also the following result:

Lemma 4 *Let $v(t)$, $t \in [0, T)$ be a weak solution of (1). Then*

$$\frac{\partial}{\partial t} E_\lambda(v(t)) \leq 0 \quad in \quad (0, T). \tag{49}$$

Proof. By the additional regularity obtained in Section 2, there exists $\frac{\partial}{\partial t}E_\lambda(v(t))$ in $(0, T)$ and

$$\frac{\partial}{\partial t}E_\lambda(v(t)) = D_u E_\lambda(v(t))(v_t(t)) =< -\Delta v(t) - \lambda|v|^{\beta-1}v + |v|^{\alpha-1}v, v_t(t) >= -||v_t(t)||^2_{L^2} \leq 0.$$

Thus we get the result. □

Thanks to Lemma 4 to get our main conclusion it is enough to prove the following result:

Lemma 5 *Let $\lambda > \lambda^{gr}$. Then there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$*

$$E_\lambda(u) > \hat{E}_\lambda \;\; \forall u \in V_\delta(\widehat{G}_\lambda) \setminus \{\widehat{G}_\lambda\}. \tag{50}$$

PROOF. If not, for every $\delta \in (0, \delta_0)$ there exists $u^\delta \in V_\delta(\widehat{G}_\lambda) \setminus \{\widehat{G}_\lambda\}$ such that $E_\lambda(u^\delta) \leq \hat{E}_\lambda$. This implies that there exists a sequence $u^n \in V_\delta(\widehat{G}_\lambda) \setminus \{\widehat{G}_\lambda\}$ such that

$$\inf_{u \in \widehat{G}_\lambda} ||u - u^n||_1 \to 0 \;\; as \;\; n \to \infty,$$

and

$$E_\lambda(u^n) \leq \hat{E}_\lambda \quad n = 1, 2, ... \tag{51}$$

Therefore that there exists a sequence $v_n \in \widehat{G}_\lambda$ such that

$$||v_n - u^n||_1 \to 0 \ \ as \ \ n \to \infty \tag{52}$$

Note that (v_n) is a minimizing sequence of (40), since $E_\lambda(v_n) \equiv \hat{E}_\lambda$ for all $n = 1, 2, \ldots$. Hence we may apply the arguments from the proof of Lemma 9 in [44] and conclude that there exists a ground state $u_\lambda \neq 0$ of (2) such that $v_n \to u_\lambda$ strongly in H^1_0. Hence by (52) we derive that $u^n \to u_\lambda$ strongly in $\mathrm{H}^1_0(\Omega)$ as $n \to \infty$. Note that the functions u^n, for any $n = 1, 2, \ldots$, are not ground states of (2). Furthermore, $r_{\min}(u_\lambda) = 1$ since $E''_\lambda(u_\lambda) > 0$. Thus

$$E_\lambda(r_{min}(u^n)u^n) > \hat{E}_\lambda \quad n = 1, 2, \ldots.$$

Moreover, this and (51) yield that

$$1 < r_{max}(u^n) < r_{\min}(u^n). \tag{53}$$

Note that $r_{max}(\cdot), r_{\min}(\cdot) : \mathrm{H}^1_0(\Omega) \to \mathbb{R}$ are continuous maps. Hence

$$r_{\min}(u^n) \to r_{\min}(u_\lambda) = 1 \ \text{ as } n \to \infty,$$

since $u^n \to u_\lambda$ in $\mathrm{H}^1_0(\Omega)$ as $n \to \infty$. Then by (53) we have also

$$r_{max}(u^n) \to r_{\min}(u_\lambda) = 1 \ \text{ as } \quad n \to \infty.$$

From this and since $E''_\lambda(r_{max}(u^n)u^n) \leq 0$ and $E''_\lambda(r_{\min}(u^n)u^n) \geq 0$ we conclude that

$$E''_\lambda(u_\lambda) = 0.$$

But this is impossible by the assumption. This contradiction completes the proof. □

PROOF OF THEOREM 1, (2^o) *Second step*. The conclusion will follow from

Lemma 6 *Let $\lambda > \lambda^{gr}$. Then for any given $\varepsilon > 0$, there exists $\delta \in (0, \delta_0)$ such that*

$$\inf_{u \in \widehat{G}_\lambda} ||u - v(t; w_0)||_1 < \varepsilon, \quad \forall w_0 \in V_\delta(\widehat{G}_\lambda) \setminus \{\widehat{G}_\lambda\}, \ \ \forall t > 0. \tag{54}$$

PROOF. Without loss of generality we may assume that $\varepsilon \in (0, \delta_0)$. Consider

$$d_\varepsilon := \inf\{E_\lambda(w) : w \in K \subset \mathrm{H}^1_0(\Omega), \ \inf_{u \in \widehat{G}_\lambda} ||u - w||_1 = \varepsilon\}. \tag{55}$$

Then $d_\varepsilon > \hat{E}_\lambda$. Indeed, assume the opposite, that there is a sequence $w^n \in K$, $\inf_{u \in \widehat{G}_\lambda} ||u - w^n||_1 = \varepsilon$ and $E_\lambda(w^n) \to \hat{E}_\lambda$. Since the set $\mathbb{G}_\lambda$ is bounded in H^1_0 it follows that (w^n) is bounded in $\mathrm{H}^1_0(\Omega)$ and therefore by the embedding theorem there exists a subsequence (again denoted by (w^n)) such that $w^n \to w_0$ weakly in $\mathrm{H}^1_0(\Omega)$ and strongly in $\mathrm{L}^p(\Omega)$, $1 \leq p < 2^*$ for some $w_0 \in \mathrm{H}^1_0(\Omega)$. Since $||u||_1^2$ is a weakly lower semi-continuous functional on $\mathrm{H}^1_0(\Omega)$, one has $\hat{E}_\lambda \geq E_\lambda(w_0)$ and $||u - w_0||_1 \leq \varepsilon$ for all $u \in \widehat{G}_\lambda$. By Lemma 5 this is possible only if $w_0 \in \widehat{G}_\lambda$. But then $\hat{E}_\lambda = E_\lambda(w_0)$ implies that $w^n \to w_0$ strongly in $\mathrm{H}^1_0(\Omega)$. From here it is not hard to show that $\varepsilon = \inf_{u \in \widehat{G}_\lambda} ||u - w^n||_1 \to \inf_{u \in \widehat{G}_\lambda} ||u - w_0||_1$. But then $w_0 \notin \widehat{G}_\lambda$. Thus we get a contradiction.

Let $\sigma > 0$ be an arbitrary value such that $d_\varepsilon - \sigma > \hat{E}_\lambda$. Then by continuity of $E_\lambda(w)$ one can find $\delta \in (0, \varepsilon)$ such that

$$E_\lambda(w) < d_\varepsilon - \sigma \quad \forall w \in V_\delta(\widehat{G}_\lambda) \subseteq V_\varepsilon(\widehat{G}_\lambda). \tag{56}$$

We claim that for any $w_0 \in V_\delta(\widehat{G}_\lambda)$ the solution $v(t, w_0)$ belongs to $V_\varepsilon(\widehat{G}_\lambda)$ for all $t > 0$. Indeed, suppose the opposite, then since $v(t, w_0) \in \mathcal{C}((0, +\infty) : \mathrm{H}^1_0(\Omega))$ there exists $t_0 > 0$ and $u_\lambda \in \widehat{G}_\lambda$ such that $||u_\lambda - v(t_0, w_0)||_1 = \varepsilon$. Then by (55),

$$d_\varepsilon \leq E_\lambda(v(t_0, w_0)).$$

On the other hand, by Lemma 4 we have $E_\lambda(v(t_0, w_0)) \leq E_\lambda(w_0)$. Thus by (56) one gets

$$d_\varepsilon \leq E_\lambda(v(t_0, w_0)) \leq E_\lambda(w_0) < d_\varepsilon - \sigma.$$

This contradiction proves the claim. □
This completes the proof of Theorem 1, (2^o).

PROOF OF THEOREM 3, (II), (2^o) It is a direct consequence of Theorem 1, (2^o) since by Lemma 1 we know that if $(\alpha, \beta) \in \mathcal{E}_s(\mathrm{N})$ then any solution u of (2) satisfies $E_\lambda''(u) > 0$.

7 Proof of Theorem 2, (2^o): globally unstability for $\beta = 1$

Let $\beta = 1$. Let us introduce the set

$$\mathcal{W} := \{u \in \mathrm{H}_0^1(\Omega) : E_\lambda(u) < \widehat{E}_\lambda, \;\; E_\lambda'(u) < 0\}, \tag{57}$$

(denoted as the *exterior potential well* in [50]). The proof of both instability results of this Section will use the following Lemma:

Lemma 7 *$\mathcal{W}$ is invariant under the flow (1).*

PROOF. Let $v(t, v_0)$ be a weak solution of (1). Then using the additional regularity obtained in Section 2 we have

$$E_\lambda(v(t)) \leq \int_0^t ||v_t||_{L^2}^2 ds + E_\lambda(v(t)) \leq E_\lambda(v_0) < \widehat{E}_\lambda.$$

for all $t > 0$. Thus $v(t)$ may leave $\mathcal{W}$ only if there is a time $t_0 > 0$ such that $r(v(t_0)) = 1$ (since, formally, $E_\lambda'(v(t_0)) = 0$). But then, by (36), we have

$$E_\lambda(v(t_0)) = \max_{r>0} E_\lambda(rv(t_0)) \geq \widehat{E}_\lambda.$$

Thus we get a contradiction and indeed

$$E_\lambda(v(t, v_0)) < \widehat{E}_\lambda, \quad E_\lambda'(v(t, v_0)) < 0 \;\; \forall t > 0 \tag{58}$$

for any $v_0 \in \mathcal{W}$. □

We have:

Lemma 8 *If $v_0 \in \mathcal{W}$, then $||v(t, v_0)||_{L^2(\Omega)} \to \infty$ as $t \to +\infty$.*

To prove this lemma we need the following result:

Proposition 12 *Assume $\beta = 1$ and that $v \in \mathrm{L}^\infty((0, +\infty); \mathrm{H}_0^1(\Omega))$. Then there exists $c_0 < 0$, which does not depend on $t > 0$ such that*

$$E_\lambda'(v(t)) \leq c_0 < 0 \quad \textit{for a.e. } t > 0. \tag{59}$$

PROOF. By regularizing $v(t, v_0)$ we can assume that $E_\lambda'(v(t))$ is continuous in t. Suppose, contrary to our claim, that there is (t_m) such that the sequence $v_m := v(t_m)$, $m = 1, 2, ...$ satisfies

$$E_\lambda'(v_m) \to 0 \;\text{ as }\; m \to \infty. \tag{60}$$

Note that by Lemma 7 we have

$$E_\lambda(v_m) < \widehat{E}_\lambda \;\text{ for }\; m = 1, 2, ... \tag{61}$$

By assumption (v_m) is bounded in $\mathrm{H}_0^1(\Omega)$. Therefore we have the following convergences (up choosing a subsequence)

$$v_m \to \bar{v} \text{ as } m \to \infty \text{ in } \mathrm{L}^p, \ 1 < p < 2^* \tag{62}$$
$$v_m \rightharpoonup \bar{v} \text{ as } m \to \infty \text{ weakly in } \mathrm{H}_0^1(\Omega) \tag{63}$$
$$\lim_{m\to\infty} E_\lambda(v_m) = a \tag{64}$$

for some $\bar{v} \in \mathrm{H}_0^1(\Omega)$ and $a \in \mathbb{R}$. Hence by the weakly lower semi-continuity of $T(u)$ in $\mathrm{H}_0^1(\Omega)$ we have

$$E_\lambda(\bar{v}) \le \lim_{m\to\infty} E_\lambda(v_m) = a \tag{65}$$
$$E'_\lambda(\bar{v}) \le \lim_{m\to\infty} E'_\lambda(v_m) = 0. \tag{66}$$

Since $v \in \mathrm{C}([0,+\infty) : \mathrm{H}_0^1(\Omega))$ by Proposition 1 we have

$$\int_0^t ||v_t||^2_{L^2} ds + E_\lambda(v(t)) \le E_\lambda(v(0)) = E_\lambda(v_0). \tag{67}$$

Hence

$$a = \lim_{m\to\infty} E_\lambda(v_m) \le E_\lambda(v_0) < \widehat{E}_\lambda$$

for any $v_0 \in \mathcal{W}$ and therefore $E_\lambda(\bar{v}) < \widehat{E}_\lambda$. Observe that this implies a contradiction in case equality holds in (66). Indeed, if $E'_\lambda(\bar{v}) = 0$, then $r(\bar{v}) = 1$ and therefore (35), (37) and (38) yield $E_\lambda(\bar{v}) \ge \widehat{E}_\lambda$. Suppose that $E'_\lambda(\bar{v}) < 0$. Then there is $r \in (0,1)$ such that $E'_\lambda(r\bar{v}) = 0$. Observe that (62) and (64) imply

$$\frac{1}{2} \lim_{m\to\infty} \mathrm{H}_\lambda(v_m) = a - \frac{1}{1+\alpha} A(\bar{v}) \tag{68}$$

and (60) implies

$$\lim_{m\to\infty} \mathrm{H}_\lambda(v_m) = -A(\bar{v}). \tag{69}$$

From here we obtain

$$\begin{aligned}
E_\lambda(r\bar{v}) &= \frac{r^2}{2} H_\lambda(\bar{v}) + \frac{r^{1+\alpha}}{1+\alpha} A(\bar{v}) \\
&\le \frac{r^2}{2} \lim_{m\to\infty} \mathrm{H}_\lambda(v_m) + \frac{r^{1+\alpha}}{1+\alpha} A(\bar{v}) \\
&= \frac{1}{2} \lim_{m\to\infty} \mathrm{H}_\lambda(v_m) + \frac{1}{2}(r^2-1) \lim_{m\to\infty} \mathrm{H}_\lambda(v_m) + \frac{r^{1+\alpha}}{1+\alpha} A(\bar{v}) \\
&= a - \frac{1}{1+\alpha} A(\bar{v}) - \frac{1}{2}(r^2-1) A(\bar{v}) + \frac{r^{1+\alpha}}{1+\alpha} A(\bar{v}) \\
&= a + \left[-\frac{1}{1+\alpha} - \frac{1}{2}(r^2-1) + \frac{r^{1+\alpha}}{1+\alpha} \right] A(\bar{v}).
\end{aligned}$$

It is easy to see that

$$\max_{0 \le r \le 1} \left\{ \left[-\frac{1}{1+\alpha} - \frac{1}{2}(r^2-1) + \frac{r^{1+\alpha}}{1+\alpha} \right] \right\} = 0.$$

Thus we get that $E_\lambda(r\bar{v}) \le a < \widehat{E}_\lambda$. However this contradicts the definition of $\hat{E}_\lambda$, since $E'_\lambda(r\bar{v}) = 0$. This completes the proof of the proposition. □

Proof of Lemma 8. Continuation. Suppose, contrary to our claim, that the set $(v(t))$, $t > 0$ is bounded in $\mathrm{L}^2(\Omega)$. Then this set is also bounded in $\mathrm{H}_0^1(\Omega)$, since $\mathrm{H}_\lambda(v(t)) := T(v(t)) - \lambda G(v(t)) < 0$ for all $t > 0$. Let us consider

$$y(t) := ||v(t)||^2_{\mathrm{L}^2}, \ \ t \ge 0,$$

where $v(t) := v(t, v_0)$. Observe that

$$||v(t)||_{L^2}^2 = ||v_0||_{L^2}^2 + 2\int_0^t (v_t(s), v(s))\, ds$$

and by (1)

$$(v_t(s), v(s)) = (\Delta v(s) + \lambda v(s) - |v(s)|^{\alpha-1}v(s), v(s)) = -E'_\lambda(v(s)).$$

Therefore

$$y(t) = ||v_0||_{L^2}^2 - 2\int_0^t E'_\lambda(v(s))ds, \tag{70}$$

and

$$\frac{d}{dt}y(t) \equiv \dot{y}(t) = -2E'_\lambda(v(t)).$$

Hence estimates (59) of Proposition 12 yield $\dot{y}(t) > -2c_0 > 0$ for all $t > 0$ and therefore $y(t) = ||v(t)||_{L^2}^2 \to +\infty$ as $t \to \infty$. This completes the proof of Lemma 8. □

PROOF OF THEOREM 2, (2^o) Let u_λ be a ground state of (1) and given any $\delta > 0$. Observe that for any $r > 1$

$$E_\lambda(ru_\lambda) < \widehat{E}_\lambda \ \text{ and } \ E'_\lambda(ru_\lambda) < 0.$$

Thus $ru_\lambda \in \mathcal{W}$ for any $r > 1$ and by Lemma 8, $||v(t; v_0)||_{L^2} \to +\infty$ with $v_0 = ru_\lambda$. Therefore

$$||u_\lambda - v(t; v_0)||_{L^2} \to +\infty \quad \text{as} \quad t \to \infty.$$

On the other hand, it is clear that $||u_\lambda - ru_\lambda||_{L^2} < \delta$ for sufficiently small $|r - 1|$. This concludes the proof of Theorem 2, (2^o). □

8 Proof of Theorem 1, (3^0) and Theorem 3, (I). Linearized unstability

The proof of Theorem 1, (3^0) will follow from

Lemma 9 *Let u_λ be a nonnegative weak solution of (2) such that $E''(u_\lambda) < 0$, then u_λ is an unstable stationary solution of (1) in the sense that $\lambda_1(-\Delta - \lambda\beta u_\lambda^{\beta-1} + \alpha u_\lambda^{\alpha-1}) < 0$.*

PROOF. Let us start with the case in which u_λ is a nonnegative flat weak solution of $SP(\alpha, \beta, \lambda)$ and Ω is a ball. Then the corresponding linearized eigenvalue problem at u_λ is

$$\begin{cases} -\Delta\psi - (\lambda\beta u_\lambda^{\beta-1} - \alpha u_\lambda^{\alpha-1})\psi = \mu\psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{71}$$

Then there is a first eigenvalue μ_1 to (71) with a positive eigenfunction $\psi_1 > 0$ such that $\psi_1 \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$. The existence of μ_1 is a particular case of the results in [29] using the estimates on the boundary behavior of u_λ obtained in [24], [25], namely that

$$\underline{K}d(x)^{2/(1-\alpha)} \leq u_\lambda(x) \leq \overline{K}d(x)^{2/(1-\alpha)} \quad \text{for any } x \in \overline{\Omega}, \tag{72}$$

for some constants $\overline{K} > \underline{K} > 0$. We shall sketch the argument for the reader's convenience. From (72) it follows that, roughly speaking $u_\lambda(x)^{\alpha-1}$ "behaves like" $d(x)^{-2}$ and $u_\lambda(x)^{\beta-1}$ as $d(x)^{-2(1-\beta)/(1-\alpha)}$ with $\gamma := 2(1-\beta)/(1-\alpha) < 2$ from $\alpha < \beta$. Then from the usual monotonicity properties of eigenvalues it is enough to show that the first eigenvalue of the problem

$$\begin{cases} -\Delta w + \dfrac{\alpha}{d(x)^2}w - \dfrac{\lambda\beta}{d(x)^\gamma}w = \mu w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{73}$$

is well-defined and has the usual properties. This is carried out by reducing the problem to an equivalent "fixed point" argument for an associated (linear) eigenvalue problem. Assume first that $\mu > 0$. Then (73) is equivalent to the existence of μ such that $\mathrm{r}(\mu) = 1$, where $\mathrm{r}(\mu)$ is the first eigenvalue for the associated problem

$$\begin{cases} -\Delta w + \dfrac{\alpha}{d(x)^2} w = \mathrm{r}\left(\dfrac{\lambda\beta}{d(x)^\gamma} + \mu\right) w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{74}$$

That $\mathrm{r}(\mu) > 0$ is well-defined follows by showing that (74) is equivalently formulated as $Tw = \mathrm{r}w$, with $T = i \circ P \circ F$, where $F : L^2(\Omega, d^\gamma) \to \mathrm{H}^{-1}(\Omega)$ defined by

$$F(w) = \frac{\lambda\beta}{d(x)^\gamma} w + \mu w,$$

$P : \mathrm{H}^{-1}(\Omega) \to \mathrm{H}_0^1(\Omega)$ is the solution operator for the linear problem

$$\begin{cases} -\Delta z + \dfrac{\alpha}{d(x)^2} z = h(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{75}$$

for $h \in \mathrm{H}^{-1}(\Omega)$, and $i : \mathrm{H}_0^1(\Omega) \to L^2(\Omega, d^\gamma)$ is the standard embedding. It is possible to prove that F and P are continuous and i is compact by using Hardy's inequality and the Lax-Milgram Lemma (see[6], [29]). Since T is an irreductible compact linear operator and applying the weak maximum principle, it is possible to apply Krein-Rutman's theorem in the formulation (weaker than usual) in [18]. We have the variational formulation

$$\mathrm{r}(\mu) = \inf_{w \in \mathrm{H}_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \left(|\nabla w|^2 + \frac{\alpha}{d(x)^2} w^2\right) dx}{\displaystyle\lambda\beta \int_\Omega \frac{w^2}{d(x)^\gamma}\, dx + \mu \int_\Omega w^2 dx}. \tag{76}$$

Hence a positive eigenvalue exists if and only if there is a $\mu > 0$ such that $\mathrm{r}(\mu) = 1$. A completely analogous argument gives the formulation for $\mu < 0$, namely with

$$\mathrm{r}_1(\mu) = \inf_{w \in \mathrm{H}_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \left(|\nabla w|^2 + \frac{\alpha}{d(x)^2} w^2 - \mu w^2\right) dx}{\displaystyle\lambda\beta \int_\Omega \frac{w^2}{d(x)^\gamma}\, dx}. \tag{77}$$

Notice that $\mathrm{r}(\mu)$ (resp. $\mathrm{r}_1(\mu)$) is decreasing (resp. increasing) in μ ($-\mu$). Then

$$\mathrm{r}(0) = \mathrm{r}_1(0) = \inf_{w \in \mathrm{H}_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \left(|\nabla w|^2 + \frac{\alpha}{d(x)^2} w^2\right) dx}{\displaystyle\lambda\beta \int_\Omega \frac{w^2}{d(x)^\gamma}\, dx},$$

and there exists a positive eigenvalue if $\mathrm{r}(0) > 1$ and a negative one if $\mathrm{r}(0) < 1$.
Coming back to our instability analysis, by Courant minimax principle we have

$$\mu_1 = \inf_{\psi \in \mathrm{H}_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \left(|\nabla\psi|^2 - (\lambda\beta u_\lambda^{\beta-1} - \alpha u_\lambda^{\alpha-1})\psi^2\right) dx}{\displaystyle\int_\Omega |\psi|^2\, dx} \tag{78}$$

Let us put $\psi = u_\lambda$ in the minimizing functional of (78). Then we get

$$\frac{\displaystyle\int_\Omega \left(|\nabla u_\lambda|^2 - (\lambda\beta u_\lambda^{\beta-1} - \alpha u_\lambda^{\alpha-1}) u_\lambda^2\right) dx}{\displaystyle\int_\Omega |u_\lambda|^2\, dx} = \frac{E''_\lambda(u_\lambda)}{\displaystyle\int_\Omega |u_\lambda|^2\, dx} < 0$$

since by the assumption $E''(u_\lambda) < 0$. This yields by (78) that $\lambda_1(-\Delta - \lambda\beta u_\lambda^{\beta-1} + \alpha u_\lambda^{\alpha-1}) := \mu_1 < 0$. Thus we get the unstability.
If u_λ is a nonnegative compactly supported weak solution of $SP(\alpha,\beta,\lambda)$ then we know that its support is a ball and on this ball u_λ becomes a flat solution to which we can apply the preceding arguments.
The same arguments still works in the case of classical $\mathrm{C}^1(\overline{\Omega}) \cap \mathrm{C}^2(\Omega)$ positive solutions $u > 0$ in Ω such that $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$. Since in this case $u_\lambda(x)$ "behaves like" $d(x)$ close to the boundary, we arrive now to the linearized problem

$$\begin{cases} -\Delta w + \dfrac{\alpha}{d(x)^{1-\alpha}} w - \dfrac{\lambda\beta}{d(x)^{1-\beta}} w = \mu w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{79}$$

and we can reason as above. The argument is actually easier since now $(0 < 1-\alpha < 1, 0 < 1-\beta < 1)$ we have, by Hardy-Sobolev inequality, $F : L^2(\Omega) \to \mathrm{H}^{-1}(\Omega)$ and i is just Rellich theorem for $\mathrm{H}_0^1(\Omega) \to L^2(\Omega)$. The rest of the reasoning follows in exactly the same way and we arrive finally to

$$\mu_1 < \frac{\displaystyle\int_\Omega \left(|\nabla u_\lambda|^2 - (\lambda\beta u_\lambda^{\beta-1} - \alpha u_\lambda^{\alpha-1}) u_\lambda^2 \right) dx}{\displaystyle\int_\Omega |u_\lambda|^2 \, dx} = \frac{E_\lambda''(u_\lambda)}{\displaystyle\int_\Omega |u_\lambda|^2 \, dx} < 0 \tag{80}$$

□

PROOF OF THEOREM 3, (I). Assume $(\alpha,\beta) \in \mathcal{E}_u(\mathrm{N})$. Let u_λ be a flat or compactly supported solution of (2). Then by Lemma 1, (2^o) we have $E_\lambda''(u_\lambda) < 0$. This yields by Lemma 9 that u_λ is a linearized unstable stationary solution of the parabolic problem (1). □

Remark 6 *Linearized stability (and unstability) results were obtained in [39] working this time in the space $C_0^1(\overline{\Omega})$ and $C^{1,\gamma}(\overline{\Omega})$ $(0 < \gamma < 1)$ spaces. It was also proved in this paper that linearized stability implies Lyapounov stability. It is reasonable to think that the same result still holds in $\mathrm{H}_0^1(\Omega)$. Interesting results in this sense for degenerate parabolic problems were obtained in [6], but working in different function spaces (see [29] for generalizations of some of these works).*

Appendix. Proof of Lemma 3: Existence of a ground state of (2) in $\mathbb{R}^N$ for $\beta = 1$

Let us consider (46), i.e.

$$\widehat{E}^\infty = \min\{J(u) : \; u \in \mathrm{H}^1(\mathbb{R}^\mathrm{N}) \setminus 0, \; \mathrm{R}(u) < 0\}. \tag{81}$$

Lemma 10 *There exists a minimizer v of (81).*

PROOF. Let (v_m) be a minimizing sequence for (81). Since $J(u)$ is a zero-homogeneous functional, we may assume that $||v_m||_1 = 1$, $m = 1,2,...,$. This implies that

$$|\mathrm{H}(v_m)| < C < \infty \text{ uniformly on } m = 1,2,.... \tag{82}$$

Observe that

$$||v_m||^2_{\mathrm{L}^2(\mathbb{R}^\mathrm{N})} \equiv \int |v_m|^2 dx > c_1 > 0 \tag{83}$$

uniformly on $m = 1,2,....$ Indeed, if we suppose the contrary, $\int |v_m|^2 dx \to 0$ as $m \to \infty$, then the assumption $||v_m||_1 = 1$, $m = 1,2,...$ implies that $\int |\nabla v_m|^2 \, dx \to 1$ and therefore $\mathrm{H}(v_m) = \int |\nabla v_m|^2 \, dx - \int |v_m|^2 \, dx \to 1$ as $m \to \infty$. But this is impossible, since by the construction $\mathrm{H}(v_m) < 0$.

Let us show that

$$A(v_m) > c_0 > 0 \text{ uniformly on } m = 1, 2, ... \tag{84}$$

Assume the opposite, that $A(v_m) \to 0$ as $m \to \infty$. Then $\int |v_m|^2 dx \to 0$ as $m \to \infty$, since by Hölder and Sobolev inequalities

$$\int |v_m|^2 dx \le \left(\int |v_m|^{\alpha+1} dx \right)^{\frac{\kappa}{\alpha+1}} \left(\int |v_m|^{2^*} dx \right)^{\frac{\alpha+1-\kappa}{\alpha+1}} \le C_0 A(v_m)^{\frac{\kappa}{\alpha+1}} ||v_m||_1^{2^* \frac{\alpha+1-\kappa}{\alpha+1}}, \tag{85}$$

where $\kappa = \frac{(\alpha+1)(2^*-2)}{2^*-(\alpha+1)}$. But this contradicts (83).
Observe that (45), (82) and (84) yield

$$\widehat{E}^\infty > 0, \tag{86}$$

and we have

$$0 < c_0 < ||v_m||^{1+\alpha}_{\mathrm{L}^{1+\alpha}} \equiv A(v_m) < C_1 < +\infty \tag{87}$$

uniformly on $m = 1, 2, \dots$
We need the following lemma [36, Lemma I.1, p.231]

Lemma 11 *Let $1 \le q < +\infty$ with $q \le 2^*$ if $\mathrm{N} \ge 3$. Assume that (w_n) is bounded in $\mathrm{H}^1(\mathbb{R}^\mathrm{N})$ and $\mathrm{L}^q(\mathbb{R}^N)$, and*

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_R} |w_n|^q dx \to 0 \;\; as \;\; n \to \infty, \;\; for\ some \;\; R > 0$$

Then $||w_n||_{\mathrm{L}^\beta} \to 0$ for $\beta \in (q, 2^*)$.

Let $R > 0$. Observe that

$$\liminf_{m\to\infty} \sup_{y\in\mathbb{R}^\mathrm{N}} \int_{y+B_R} |v_m|^{1+\alpha} dx := \delta > 0. \tag{88}$$

Indeed, let us assume that

$$\liminf_{m\to\infty} \sup_{y\in\mathbb{R}^\mathrm{N}} \int_{y+B_R} |v_m|^{1+\alpha} dx = 0.$$

Then by Lemma 11 we have $||v_m||_{\mathrm{L}^2} \to 0$ as $m \to \infty$. But this contradicts (84).
Thus there is a sequence $\{y_m\} \subset \mathbb{R}^\mathrm{N}$ such that

$$\int_{y_m+B_R} |v_m|^{1+\alpha} dx > \frac{\delta}{2}, \quad m = 1, 2, ...$$

Introduce $u_m := v_m(\cdot + y_m)$, $m = 1, 2, ...$ Then

$$\int_{B_R} |u_m|^{1+\alpha} dx > \frac{\delta}{2}, \quad m = 1, 2, ..., \tag{89}$$

and $\{u_m\}$ is a minimizing sequence of (81).
Furthermore, by the zero-homogeneity of $J(u)$ now we may normalize the sequence $\{u_m\}$ (again denoted by $\{u_m\}$) such that

$$A(u_m) = 1, \quad m = 1, 2, ... \tag{90}$$

Then (85) and the assumption $H(u_m) < 0$, $m = 1, 2, ...$, imply that the renormalized sequence $\{u_m\}$ will be again bounded in $\mathrm{H}^1(\mathbb{R}^\mathrm{N})$. Thus by Eberlein-Smulian theorem there is a subsequence of $\{u_m\}$ (again denoting $\{u_m\}$) and a limit point $\bar{u} \in \mathrm{H}^1(\mathbb{R}^\mathrm{N})$ such that

$$u_m \rightharpoonup \bar{u} \text{ weakly in } \mathrm{H}^1(\mathbb{R}^\mathrm{N}) \text{ as } m \to \infty. \tag{91}$$

Furthermore

$$u_m \to \bar{u} \text{ a.e. on } \mathbb{R}^\mathrm{N} \text{ as } m \to \infty, \tag{92}$$

and for $2 < q < 2^*$

$$u_m \to \bar{u} \text{ in } \mathrm{L}^q_{loc} \text{ as } m \to \infty, \tag{93}$$

since by Rellich-Kondrachov theorem $\mathrm{H}^1_0(B_R)$ is compactly embedded in $\mathrm{L}^q(B_R)$ for $2 < q < 2^*$ and any $B_R := \{x \in \mathbb{R}^\mathrm{N} : |x| \leq R\}$, $R > 0$. Note that (89) implies that

$$\bar{u} \neq 0.$$

We need the Brezis-Lieb lemma [10]:

Lemma 12 *Let Ω be an open subset of $\mathbb{R}^\mathrm{N}$ and let $\{w_n\} \subset \mathrm{L}^q(\Omega)$, $1 \leq q < \infty$. If*

a) *$\{w_n\}$ bounded in $\mathrm{L}^q(\Omega)$,*

b) *$w_n \to w$ a.e. on Ω, then*

$$\lim_{n\to\infty} \left(||w_n||^q_{\mathrm{L}^q} - ||w_n - w||^q_{\mathrm{L}^q}\right) = ||w||^q_{\mathrm{L}^q}.$$

Let us denote $\omega_m := u_m - \bar{u}$. Then Brezis-Lieb lemma yields

$$1 = A(\bar{u}) + \lim_{m\to\infty} A(\omega_m). \tag{94}$$

Let us remark that $a := \lim_{m\to\infty} \mathrm{H}(u_m) < 0$ since otherwise

$$J(u_m) = \frac{(1-\alpha)}{2(1+\alpha)} \frac{1}{(-\mathrm{H}(u_m))^{\frac{1+\alpha}{1-\alpha}}} \to +\infty$$

Observe

$$\lim_{m\to\infty} \mathrm{H}(\omega_m) = \lim_{m\to\infty} \mathrm{H}(u_m) - \mathrm{H}(\bar{u}). \tag{95}$$

Suppose for the contrary that $\lim_{m\to\infty} \mathrm{H}(\omega_m) > 0$. Then $\lim_{m\to\infty} \mathrm{H}(u_m) - \mathrm{H}(\bar{u}) > 0$ and therefore $0 > a > \mathrm{H}(\bar{u})$. Hence $\mathrm{H}(\bar{u}) < 0$. Notice

$$\mathrm{H}(\omega_m) = \mathrm{H}(\bar{u}) + \mathrm{H}(u_m) - \mathrm{H}'(u_m)(\bar{u}). \tag{96}$$

However due to weak convergence (91) we have $H'(\omega_m)(u) \to 0$ as $m \to \infty$ and therefore $H(\omega_m) < 0$ for sufficiently large m, since $H(\bar{u}) < 0$ and $H(u_m) < 0$ for $m = 1, 2, ...$ Thus we get a contradiction and indeed $\mathrm{H}(\omega_m) < 0$ for sufficiently large m.

Observe that (81) and (45) imply that for any $v \in \mathrm{H}^1(\mathbb{R}^\mathrm{N}) \setminus 0$ such that $\mathrm{H}(v) < 0$ it holds

$$-\mathrm{H}(v) \leq k_\alpha \frac{A(v)^{\frac{2}{1+\alpha}}}{(\hat{E}^\infty)^{\frac{1+\alpha}{1-\alpha}}} \tag{97}$$

where

$$k_\alpha = \left(\frac{(1-\alpha)}{2(1+\alpha)}\right)^{\frac{1-\alpha}{1+\alpha}}.$$

Assume first that $\mathrm{H}(\bar{u}) < 0$. Then

$$-\mathrm{H}(\bar{u}) \leq k_\alpha \frac{A(\bar{u})^{\frac{2}{1+\alpha}}}{(\hat{E}^\infty)^{\frac{1+\alpha}{1-\alpha}}}$$

and

$$-\mathrm{H}(\omega_m) \leq k_\alpha \frac{A(\omega_m)^{\frac{2}{1+\alpha}}}{(\hat{E}^\infty)^{\frac{1+\alpha}{1-\alpha}}}, \tag{98}$$

for sufficient large m. Since $A(u_m) = 1$, we have

$$\lim_{m\to\infty} k_\alpha \frac{1}{(-\mathrm{H}(u_m))} = (\hat{E}^\infty)^{\frac{1+\alpha}{1-\alpha}}.$$

Hence we obtain

$$\begin{aligned}
k_\alpha \frac{1}{(\hat{E}^\infty)^{\frac{1+\alpha}{1-\alpha}}} &= \lim_{m\to\infty}(-\mathrm{H}(u_m)) \\
&= -\mathrm{H}(\bar{u}) + \lim_{m\to\infty}(-\mathrm{H}(\omega_m)) \\
&\leq k_\alpha \frac{A(\bar{u})^{\frac{2}{1+\alpha}}}{(\hat{E}^\infty)^{\frac{1+\alpha}{1-\alpha}}} + \lim_{m\to\infty} k_\alpha \frac{A(\omega_m)^{\frac{2}{1+\alpha}}}{(\hat{E}^\infty)^{\frac{1+\alpha}{1-\alpha}}} \\
&= k_\alpha \frac{1}{(\hat{E}^\infty)^{\frac{1+\alpha}{1-\alpha}}} \left(A(\bar{u})^{\frac{2}{1+\alpha}} + (1 - A(\bar{u}))^{\frac{2}{1+\alpha}} \right).
\end{aligned}$$

Note since $\frac{2}{1+\alpha} > 1$, then $f(r) := r^{\frac{2}{1+\alpha}} + (1-r)^{\frac{2}{1+\alpha}} \geq 1$ for $r \in [0,1]$ and $f(r) = 1$ if and only if $r = 0$ or $r = 1$. Thus we have

$$A(\bar{u}) = 1 \ \text{ or } \ A(\bar{u}) = 0.$$

Now taking into account that $\bar{u} \neq 0$ we get that $A(\bar{u}) = 1$. Assume now that $H(\bar{u}) \geq 0$. Then as above we have

$$\begin{aligned}
k_\alpha \frac{1}{(\hat{E}^\infty)^{\frac{1+\alpha}{1-\alpha}}} &= \lim_{m\to\infty}(-\mathrm{H}(u_m)) \\
&= -\mathrm{H}(\bar{u}) + \lim_{m\to\infty}(-\mathrm{H}(\omega_m)) \\
&\leq -\mathrm{H}(\bar{u}) + \lim_{m\to\infty} k_\alpha \frac{A(\omega_m)^{\frac{2}{1+\alpha}}}{(\hat{E}^\infty)^{\frac{1+\alpha}{1-\alpha}}} \\
&= -\mathrm{H}(\bar{u}) + k_\alpha \frac{1}{(\hat{E}^\infty)^{\frac{1+\alpha}{1-\alpha}}} \left((1 - A(\bar{u}))^{\frac{2}{1+\alpha}} \right) \\
&\leq k_\alpha \frac{1}{(\hat{E}^\infty)^{\frac{1+\alpha}{1-\alpha}}} \left((1 - A(\bar{u}))^{\frac{2}{1+\alpha}} \right).
\end{aligned}$$

But this inequality implies that $A(\bar{u}) = 0$. Now taking into account that $\bar{u} \neq 0$ we get a contradiction.

Hence by (94) we obtain $A(\omega_m) \to 0$ as $m \to \infty$ and consequently by (98) we have $(-\mathrm{H}(\omega_m)) \to 0$ as $m \to \infty$. From here it is not hard to conclude that $u_m \to \bar{u}$ strongly in $\mathrm{H}^1(\mathbb{R}^\mathrm{N})$ and therefore $J(\bar{u}) = \widehat{E}^\infty$. Thus $\bar{u}$ is a minimizer of (81). □

PROOF OF LEMMA 3. By Lemma 10 there exists a minimizer $\bar{u}$ of (81). Since J is an even functional then $|\bar{u}|$ is also a minimizer of (81). Thus we may assume that $\overline{u}$ is nonnegative function. By Proposition 8 it follows that $u = r(\bar{u})\bar{u}$ is a weak solution of (43) which is nonnegative since $r(\bar{u}) > 0$. By regularity theory we derive that $u \in \mathrm{C}^2(\mathbb{R}^\mathrm{N})$. □

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J.I. Díaz
Instituto de Matemática Interdisciplinar
Universidad Complutense de Madrid
28040 Madrid, Spain
jidiaz@ucm.es

J. Hernández
Departamento de Matemáticas
Universidad Autónoma de Madrid
28049 Cantoblanco, Madrid, Spain
jesus.hernandez@uam.es

Y. Il'yasov
Institute of Mathematics
Ufa Science Center of RAS
Ufa 450077, Russia
ilyasov02@gmail.com