Decaying to zero bifurcation solution curve for some sublinear elliptic eigenvalue type problems.

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Abstract. By using some rearrangements technics, jointly with a study of the radially symmetric case, we prove that the bifurcation curve for some sublinear elliptic eigenvalue type problems converges to zero when the eigenvalue parameter λ converges to infinity. The result improves some previous work in the literature (dealing mainly with the one-dimensional case or imposing some restrictions to the spatial dimension of the open set) and has application in several contexts.

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1 Introduction

We study in this paper the bifurcation diagram, with respect to λ, of non-negative solutions to the semilinear elliptic equation

$$P(q,\alpha,\lambda) = \begin{cases} -\Delta u + q(x)|u|^{\alpha-1}u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 1$), λ is a real parameter, $0 < \alpha < 1$ and $q(x) \geq 0$ is a real function satisfying that there exists q_0 and $\overline{q}$, with $0 < q_0 \leq \overline{q}$, such that

$$q_0 \leq q(x) \leq \overline{q} \quad \text{a.e. } x \in \Omega. \tag{2}$$

Notice that, obviously, $u \equiv 0$ is always a solution of problem (1) and so our interest relies on the existence of nontrivial solutions of the problem.

One of the main motivations for studying problem $P(q,\alpha,\lambda)$, with $0 < \alpha < 1$, arises from some previous results ([15]) concerning solutions of the linear elliptic Schroedinger equation associated to the so-called *infinite well potential*. For instance, in the one-dimensional case, we can formulate it as

$$\begin{cases} -u'' + V(x)u = \lambda u & \text{in } (-R,R), \\ u(\pm R) = 0, \end{cases} \tag{3}$$

for a given $R > 0$, or on the real line, $R = +\infty$ (and thus it is assumed that $V(x) = +\infty$ if $x \notin (-R_0, R_0)$ for some $R_0 > 0$). It turns out that there is some ambiguity in the treatment of the case of the real line: what is mentioned to be "the solutions" in most of the text-books are not classical solutions of the problem since Dirac's deltas appear at $x = \pm R$. Such functions are solutions merely in the sense of distributions, but besides that, they satisfy a different equation where deltas at $x = \pm R$ must be included.

In this situation solutions of the semilinear equation $P(q,\alpha,\lambda)$ provide some kind of, say, "alternative approach" if we assume that $V(x) := q(x)|u|^{\alpha-1}$ (see a detailed analysis in [15]).

The one-dimensional case of $P(q,\alpha,\lambda)$, with $\Omega = (-R,R)$ and $q \equiv 1$, was studied in [17] by using some phase plane methods in ODEs. There it was proved that for $0 < \lambda < \lambda_1$ there is no solution to (3) and for $\lambda_1 < \lambda < \lambda^*$, where λ^* is a critical value given explicitly, there is a unique solution $u_\lambda > 0$ with $\partial u_\lambda/\partial n(\pm R) < 0$ bifurcating at infinity for $\lambda = \lambda_1$. Moreover u_λ is decreasing as a function of λ and the solution $u_{\lambda^*} > 0$ in $(-R,R)$ is such that $u'_{\lambda^*}(\pm R) = 0$: this corresponds to the definition of a *flat solution.* Moreover, from this *flat solution* solution it is possible to build continua of compact support solutions. We also obtained in [17] the asymptotic behaviour (in λ) of the solutions when $\lambda \to +\infty$; namely

$$\|u_\lambda\|_{L^\infty(-R,R)} \leq \frac{C}{\lambda^{1/(1-\alpha)}}. \tag{4}$$

Notice that this asymptotic behavior of the bifurcation curve, in the plane $\mathbb{R}^2$, with variables λ and $\|u_\lambda\|_{L^\infty(-R,R)}$, is quite exceptional in contrast to many other nonlinear problems exhibiting bifurcation phenomena: in many cases it can be shown that the corresponding solutions u_λ satisfy that $\|u_\lambda\|_{L^\infty} \nearrow +\infty$ when $\lambda \to +\infty$.

Coming back again to the consideration of the Schroedinger equation, the results of [17] imply that $\lambda^* > 0$ and $u_{\lambda^*} > 0$ are, respectively, the first eigenvalue and its corresponding eigenfunction for the linear eigenvalue problem of Schroedinger type

$$\begin{cases} -w'' + |u_{\lambda^*}|^{\alpha-1} w = \lambda w & \text{in } (-R,R), \\ w(\pm R) = 0. \end{cases} \tag{5}$$

Moreover $u'_{\lambda^*}(\pm R) = 0$ (which allows to prolongate this function by zero to the rest of $\mathbb{R}$ without generating any Dirac delta at the points $x = \pm R$). It is in this sense that we have an "alternative approach" for solutions to linear Schrödinger equation for such type of singular potentials $V(x)$.

Obviously, problem $P(q,\alpha,\lambda)$ also arises in many other contexts since it is a typical example of the so-called diffusion-reaction equations (see, e.g. [32], [33], [11]). If $q \equiv 1$ and $\alpha > 1$ it is the well-known logistic equation in population dynamics. In this case there is a unique positive solution u_λ for any $\lambda > \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian with Dirichlet boundary conditions and it follows immediately from the Strong Maximum Principle that if $u_\lambda \geq 0$, $u_\lambda \not\equiv 0$, is a solution then $u_\lambda > 0$ in Ω and $\frac{\partial u_\lambda}{\partial \nu} < 0$ on $\partial\Omega$, i.e., all non-negative solutions to the logistic equation are actually positive. But this is not the case for $P(q,\alpha,\lambda)$ for $q \equiv 1$ and $0 < \alpha < 1$ and for some problems of this kind previously studied. These one-dimensonal results were extended in [21] to the case of the p-Laplacian as well as possible singular zero order terms $-1 < \alpha < p-1$.

Problem $P(q,\alpha,\lambda)$ is studied in [24] for $q = 1$ and $0 < \alpha < 1$ as a particular case of a much more general class of problems allowing more general nonlinear terms and boundary conditions. The main result in [24] is the existence of an unbounded continuum of non-negative solutions bifurcating from infinity at the asymptotic bifurcation point λ_1. The method of proof was to apply global asymptotic bifurcation theorems by Rabinowitz [30] by using as a tool some theorem in [10]. Many other references on related problems can be found in the list of references of [18].

Existence of a bounded weak non-negative solutions for any $\lambda > \lambda_1$ was obtained later by Porretta [29] (also for $q \equiv 1$ but for a more general linear elliptic second order operator). He proves that

$$\lim_{\lambda \to +\infty} \|u_\lambda\|_{L^\infty(\Omega)} = 0$$

under the assumption

$$\frac{N-2}{N+2} < \alpha < 1.$$

He used some variational methods, namely a variant of the Ambrosetti-Rabinowitz Mountain Pass Theorem. Some complementary results concerning, for example, several estimates for some norms of the solutions are also obtained there.

A more general set of results (in the N-dimensional case) was presented in the paper [18]: by using some variational methods (Nehari manifolds), it was proved there that for any $\lambda > \lambda_1$ there exists (at least) a non-negative solution. These solutions bifurcate from infinity at λ_1 and generate an unbounded continuum of non-negative solutions. In this paper we proved a *Pohozaev type identity* showing the relevance of the coefficient $q(x)$ concerning existence (or not) of a *flat solutions* (satisfying that $u_\lambda > 0$ in Ω and $\frac{\partial u_\lambda}{\partial \nu} = 0$ on $\partial\Omega$) and *compact support solutions*. The existence of solutions with compact support in Ω was considered in Section 5 of [18]. With the usual philosophy of reaction-diffusion equations giving rise to a free boundary (see Section 1.2 of [11]), the results of the mentioned paper show that, in the case of problem (1), the "diffusion-absorption balance" condition on the nonlinearities is satisfied (since $\alpha < 1$) and that the "balance condition between the data and the domain", necessary for the formation of compact support solutions (and thus of flat solutions) is here represented by means of the requirement of assuming λ large enough (see also [23]). It is important to remark that the existence of solutions with compact support in [18] was obtained by using, in a crucial way, the assumption that $\lim_{\lambda\to+\infty} \|u_\lambda\|_\infty = 0$ (that was also used in the proof of a similar result in [23]). So, once again, decay estimates of the type (4) are relevant for different purposes. The stability of solutions was considered in [19].

The main purpose of this paper is to get some asymptotic estimates on the bifurcation curve for $\lambda \to +\infty$, similar to those obtained (4) for the one-dimensional case and $q \equiv 1$, improving the results of [29] and [18]. We shall show that no additional condition on α (or N) is needed.

A very rough argument, without any rigorous justification, to expect such estimate on the bifurcation solutions curve could come (according this author) from the rough idea that for many purposes the Laplacian operator $-\Delta u$ can be substituted by the zero order expression $\lambda_1 u$, where λ_1 is the first eigenvalue of the Laplacian with Dirichlet boundary conditions for this domain Ω. In this way, if we assume that $q(x) \equiv q_0$ we “could arrive to the approximated identity“ $\lambda_1 u + q_0 u^\alpha \approx \lambda u$. From here we can see that

$$u \approx \frac{C}{(\lambda-\lambda_1)^{1/(1-\alpha)}} \text{ with } C = \frac{1}{q_0^{1/(1-\alpha)}}.$$

This also indicates that if $\lambda < \lambda_1$ no positive solution must be possible. In fact, in Porretta [29], as an intermediate step in his proof the author proves that, for any $0 < \alpha < 1$

$$\|u_\lambda\|_{L^{\alpha+1}(\Omega)} \leq \frac{C}{\lambda^{1/(1-\alpha)}}. \tag{6}$$

The following theorem shows that this rough idea can be made precise, by some arguments which are far from trivial and make precise the correct decay estimate on the bifurcation solutions curve. Since, in contrast with the onedimensional case and $q(x) = q_0$, we can not ensure the uniqueness of weak solutions of $P(q, \alpha, \lambda)$ it is opportune to mention that our result will be valid also for a concrete subclass of weak solutions; the so called *ground state* solution, i.e. a weak solution u_λ of $P(q, \alpha, \lambda)$ which satisfies

$$E_\lambda(u_\lambda) \leq E_\lambda(w_\lambda)$$

for any other nonzero weak solution w_λ of $P(q, \alpha, \lambda)$. Here $E_\lambda(u)$ is the energy functional corresponding to $P(q, \alpha, \lambda)$ which is defined on the Sobolev space $H_0^1(\Omega)$ as follows

$$E_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{\alpha+1}\int_\Omega q(x)|u|^{\alpha+1}\, dx - \frac{\lambda}{2}\int_\Omega |u|^2\, dx.$$

Theorem 1. *Let $q(x)$ satisfying (2). Let*

$$\lambda^\# = \left(\frac{|\Omega|}{\omega_N}\right)^{2N} \tag{7}$$

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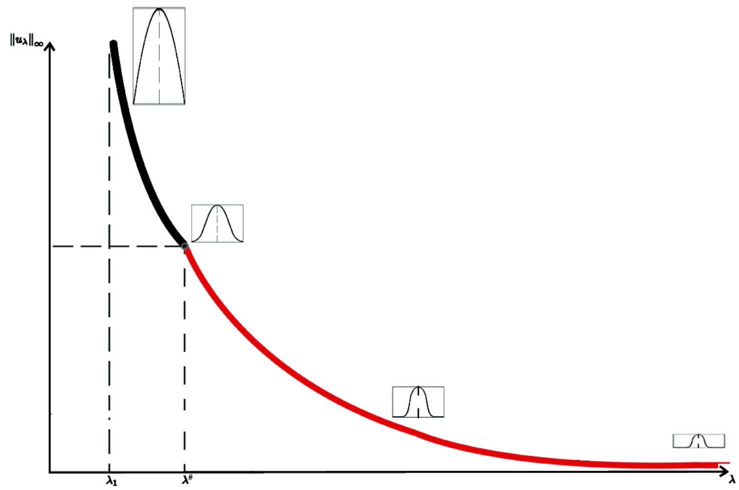

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Figure 1: Bifurcation curve for $q(x)$ constant.

with $\omega_N := |B_1(0)|$. *Then, for any* $\lambda > \lambda^{\#}$, *there exists a weak solution* u_λ *of problem* $P(q, \alpha, \lambda)$ *such that*

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq \frac{C}{\lambda^{\frac{1}{1-\alpha}}} \tag{8}$$

for some $C > 0$ *depending only on* $|\Omega|$ *and* q_0. *In addition the above weak solutions* u_λ *may be assumed to be ground solutions and, in any case, they have compact support.*

We point out that in the case $q(x) = q_0$ it was proved in [27] that any possible weak solution of $P(q, \alpha, \lambda)$ must be radially symmetric with respect to some point $x_0 \in \Omega$. Obviously this is not necessarily true for the case of a general coefficient $q(x)$ satisfying (2) and so the last conclusion of Theorem 1 seems to be new in the literature (the results of [33] require a change of sign in $q(x)$). In some sense, estimate (8) shows that there is a continuous curve bifurcating from the infinity but now from the "point" $(+\infty, 0)$. We also recall that according Proposition 2.2 of [18], the bifurcation curve $(\lambda, \|u_\lambda\|_{L^\infty(\Omega)})$ of *ground solutions* bifurcates from the infinity but now from the "point" $(\lambda_1, +\infty)$,when $\lambda \searrow \lambda_1$ (as in the one-dimensional case and $q(x) = q_0$).

Theorem 1 gives answer to the following question arising in the controllability of elastic vibrations of a body Ω: assume the open bounded set Ω vibrating (unidirectionally) in its first natural vibration mode and with a very large amplitude. So, let $v_A(x)$ satisfying

$$\begin{cases} -\Delta v = \lambda_1 v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\|v_A\|_{L^\infty(\Omega)} = A$ and A very large. Given $\epsilon > 0$ we want to control the vibration (i.e. to stabilize the vibration to the static rest) of the body, i.e., connecting the vibrational state v_A with the total equilibrium $v \equiv 0$, by means of unidirectional forces $f_\lambda(x)$, of a decreasing amplitude, $\|f_\lambda\|_{L^\infty(\Omega)} \searrow 0$, as the parameter $\lambda \nearrow +\infty$ such that if w_λ is the state associated to the control $f_\lambda(x)$

$$\begin{cases} -\Delta w = \lambda_1 w + f_\lambda(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

then we want to have

$$A - \|w_\lambda\|_{L^\infty(\Omega)} \leq \epsilon \text{ if } \lambda - \lambda_1 \text{ is small enough}$$

and

$$\|w_\lambda\|_{L^\infty(\Omega)} \leq \epsilon \text{ if } \lambda \text{ is large enough.}$$

Questions of this type can be understood as a peculiar approximate controllability property (see, e.g., the monograph [7]). Given $0 < \alpha < 1$, by taking the feedback control $f_\lambda(x) := (\lambda - \lambda_1)w(x) - |w(x)|^{\alpha-1}w(x)$ and applying Theorem 1 we see that estimate (8) shows that $\|f_\lambda\|_{L^\infty(\Omega)} \searrow 0$ as $\lambda \nearrow +\infty$.

Corollary 1 *It is possible to stabilize a very large vibration v_A of an elastic body Ω, in its first natural vibration mode, by means of a family of feedback controls $f_\lambda(x)$ of decreasing to zero amplitude. Moreover the spatial action of the controls family f_λ is a decreasing family of compact subsets of the body.*

2 Proof of the main result

The main idea of the proof will be to reduce the derivation of the estimate (8) to the simple case of radially symmetric solutions and then to extend the conclusion to the case of general domains by using rearrangement techniques. So, the first step is the consideration of a simpler formulation. Inspired in the proof of Proposition 5.1 of [18] we have:

Proposition 1. *Let $q(x) \equiv q_0 > 0$. Then, if λ^* is given by (7) the conclusion of Theorem 1 holds for the radially symmetric solution of $P(q, \alpha, \lambda)$ on $\Omega = B_R(0)$, for any $R > R^*$ with*

$$R^* = \left(\frac{\omega_N}{|\Omega|}\right)^N. \tag{9}$$

Proof. We make the change of variables

$$u_\lambda(x) = \left(\frac{q_0}{\lambda}\right)^{\frac{1}{1-\alpha}} U(\sqrt{\lambda}x) \tag{10}$$

with U solution of the special problem $P(1, \alpha, 1)$ on $\Omega = B_R(0)$, i.e.

$$\begin{cases} -\Delta u + |u|^{\alpha-1}u = u & \text{in } B_{\sqrt{\lambda}R}(0), \\ u = 0 & \text{on } \partial B_{\sqrt{\lambda}R}(0). \end{cases} \tag{11}$$

Then, if $R = R^*$, since $1/R^{*2} = \lambda^\#$ the transformed problem $P(1, \alpha, 1)$ takes place on the ball $B_1(0)$ and thus by the results of [26] we know the existence of a unique radially symmetric flat solution $u_{\lambda^*}(x) = u_{\lambda^*}(|x|)$. Since this solution can be extended to the whole $\mathbb{R}^N$ by zero outside $B_{R^*}(0)$, for any $\lambda > \lambda^\#$ we can introduce a change of variables leading to the relation

$$u_\lambda(x) = \left(\frac{\lambda^\#}{\lambda}\right)^{\frac{1}{1-\alpha}} u_{\lambda^*}\left(\frac{\sqrt{\lambda}}{\sqrt{\lambda^\#}}x\right),$$

and thus

$$\|u_\lambda\|_\infty = \left(\frac{\lambda^\#}{\lambda}\right)^{\frac{1}{1-\alpha}} \|u_{\lambda^\#}\|_\infty,$$

which leads to the estimate (8). Moreover, for any $x_0 \in B_R(0)$ such that $B_{\frac{\sqrt{\lambda^\#}}{\sqrt{\lambda}}R}(x_0) \subset B_R(0)$ we can construct the solution

$$u_\lambda(x; x_0) := \begin{cases} \left(\frac{\lambda^\#}{\lambda}\right)^{\frac{1}{1-\alpha}} u_{\lambda^\#}\left(\frac{\sqrt{\lambda}}{\sqrt{\lambda^\#}}|x - x_0|\right) & \text{on } B_{\frac{\sqrt{\lambda^\#}}{\sqrt{\lambda}}R}(x_0), \\ 0 & \text{on } B_R(0) \setminus B_{\frac{\sqrt{\lambda^\#}}{\sqrt{\lambda}}R}(x_0), \end{cases}$$

which also satisfies the estimate.■

Now we pass to the consideration of a general bounded domain Ω.

Proof of Theorem1. 1^{st} step. We assume for the moment that $q \in W^{1,1}(\Omega)$ and satisfies (2). Let $q^{\#}(|x|)$ be the radially symmetric increasing rearrangement of $q(x)$ (see, e.g. [11], [12] and its many references). Then we know that $q^{\#} \in W^{1,1}(B_R(0))$ and that

$$q_0 \le q^{\#}(|x|) \le \overline{q} \quad \text{a.e. } x \in B_R(0), \tag{12}$$

where $B_R(0)$ (which we can denote by Ω^*) is taken such that $|B_R(0)| = |\Omega|$, i.e.

$$R = \left(\frac{|\Omega|}{\omega_N}\right)^{1/N}.$$

Given a radially symmetric positive coefficient function $Q(|x|)$ we define the general set of radially symmetric problems

$$P(Q, B_R) = \begin{cases} -\Delta u + Q(|x|)|u|^{\alpha-1}u = \lambda u & \text{in } B_R(0), \\ u = 0 & \text{on } \partial B_R(0). \end{cases} \tag{13}$$

Let $\underline{U}_\lambda$ and $\overline{U}_\lambda$ be the (unique) corresponding radially symmetric solutions of $P(\overline{q}, B_{R^*})$ and $P(q_0, B_{R^*})$, respectively, given in the proof of the above proposition, for $\lambda = \lambda^* := 1/R^{*2} = \lambda^{\#}$. Since both flat solutions are positive and radially symmetric, arguing as in [26], we deduce that

$$\underline{U}_\lambda(|x|) \le \overline{U}_\lambda(|x|) \text{ for any } x \in B_{R^*}(0).$$

Moreover, thanks to (12) we know that $\underline{U}_\lambda$ (respectively $\overline{U}_\lambda$) is a subsolution (resp. a supersolution) to problem $P(q^{\#}, B_{R^*})$. Then, by the iterative method of super and sub-solutions we know that there exists the minimal and maximal associate solutions $\underline{U}_\lambda^\sharp(|x|)$, $\overline{U}_\lambda^\sharp(|x|)$ of problem $P(q^{\#}, B_{R^*})$ such that, for any other weak solution such that $\underline{U}_\lambda(|x|) \le U(x) \le \overline{U}_\lambda(|x|)$ we must have

$$\underline{U}_\lambda(|x|) \le \underline{U}_\lambda^\sharp(|x|) \le U(x) \le \overline{U}_\lambda^\sharp(|x|) \le \overline{U}_\lambda(|x|) \text{ for any } x \in B_{R^*}(0).$$

Moreover, we know that $\underline{U}_\lambda^\sharp(|x|)$ and $\overline{U}_\lambda^\sharp(|x|)$ can be assumed to be ground solutions (see, e.g. [34], p.17). As in the proof of the above proposition we can construct solutions with compact support for any $\lambda \ge \lambda^*$.
2^{nd} step. Let u_λ be any weak solution u_λ of problem $P(q, \alpha, \lambda)$ and let u_λ^* be the *symmetric decreasing rearrangement of* u_λ (see, e.g. [11] and its many references) by

$$u_\lambda^* : \Omega^* \to \mathbb{R}, \text{ with } u_\lambda^*(x) := \widetilde{u}_\lambda(\omega_N |x|^N),$$

with $\widetilde{u}_\lambda$ the *scalar decreasing rearrangement of* u_λ. At least formally, $\widetilde{u}_\lambda$ is given as the inverse function of the *distribution function μ_λ of u_λ*

$$\mu_\lambda : \mathbb{R} \to \mathbb{R}, \ \mu_\lambda(\theta) := |\{x \in \Omega : u_\lambda(x) > \theta\}|,$$

([11]). We define, $\widetilde{\Omega} := (0, |\Omega|)$ and for any $s \in [0, |\Omega|]$ the function

$$k(s) = \int_0^s \widehat{q}(\sigma)\widetilde{u}(\sigma)^\alpha d\sigma,$$

where $\widehat{q}(s)$ is the *scalar increasing rearrangement of* $q(x)$. Notice that

$$\widetilde{u}(s) = \gamma\left(\frac{1}{\widehat{q}(s)}\frac{dk}{ds}(s)\right) \quad \text{a.e. } s \in (0, |\Omega|),$$

where

$$\gamma(r) = |r|^{1/\alpha}\, sign(r).$$

It can be proved that $k(t,s)$ verifies (in a weak sense) the inequality

$$-a(s)\frac{d}{ds}\gamma(\frac{1}{\widehat{q}(s)}\frac{dk}{ds}(s)) + k(s) \leq \gamma(\frac{1}{\widehat{q}(s)}\frac{dk}{ds}(s)) \text{ on } (0,|\Omega|),$$

with

$$a(s) := \left(N\omega_N^{1/N} s^{(N-1)/N}\right)^2.$$

The proof of this inequality is quite long and technical but it is an obvious adaptation to the present framework of Theorem 1 of [12] (see also Theorem 2 of [13] and Theorem 1 of [14]).

3^rd^ step. The scalar decreasing rearrangement $\widetilde{\overline{U}_\lambda^\sharp}(s)$ of the radially symmetric maximal solution $\overline{U}_\lambda^\sharp(|x|)$ of problem $P(q^\#, B_R)$ gives rise to a function $K(s)$ defined by

$$K(s) = \int_0^s \widehat{q}(\sigma)\widetilde{\overline{U}_\lambda^\sharp}(\sigma)^\alpha d\sigma,$$

which satisfies

$$-a(s)\frac{d}{ds}\gamma(\frac{1}{\widehat{q}(s)}\frac{dK}{ds}(s)) + K(s) = \gamma(\frac{1}{\widehat{q}(s)}\frac{dK}{ds}(s)) \text{ on } (0,|\Omega|).$$

The proof of this is again an obvious adaptation to our framework of the similar results of the above mentioned references. Moreover, in both cases, we have the boundary conditions

$$\begin{cases} k(0) = K(0) = 0, \\ \frac{dk}{ds}(|\Omega|) = \frac{dK}{ds}(|\Omega|) = 0. \end{cases} \tag{14}$$

4^rd^ step. By using the T-accretiveness in $L^\infty(\widetilde{\Omega})$ of the operator

$$D(A) = \{w \in L^\infty(\widetilde{\Omega}) : w(0) = 0,\, \tfrac{dw}{ds}(|\Omega|) = 0,\, a(s)\tfrac{d}{ds}\gamma(\tfrac{1}{\widehat{q}(s)}\tfrac{dw}{ds}(s)) \in L^\infty(\widetilde{\Omega})\}$$
$$Aw = -a(s)\tfrac{d}{ds}\gamma(\tfrac{1}{\widehat{q}(s)}\tfrac{dw}{ds}(s)) \text{ if } w \in D(A),$$

we get the comparison result

$$k(s) \leq K(s) \text{ for any } s \in [0,|\Omega|]. \tag{15}$$

Although this comparison can be proved by using the techniques of [3] we shall follow here a different method which uses the T-accretiveness in $L^\infty(\widetilde{\Omega})$ of operator A (a well-known result in the previous literature; see, e.g. [4], [8]). We know that $\frac{dk}{ds}, \frac{dK}{ds} \in C^1(\delta,|\Omega|)$ for any $\delta > 0$. Then, we can reason as in [9] by using the interior semiproduc on $L^\infty(\widetilde{\Omega})$

$$\tau(z,y) = \lim_{\varepsilon \searrow 0} ess\sup\{(sign z(s))y(s) : s \in J(z,\varepsilon)\}$$

where

$$J(z,\varepsilon) = \{s \in \widetilde{\Omega} : \ |z(s)| > \|z\|_{L^\infty(\widetilde{\Omega})} - \varepsilon\}.$$

Thus, multiplying (by means of τ) the difference of the differential expressions for k and K by

$$z(s) = [k(s) - K(s)]_+,$$

if we assume that $z(s) > 0$ we get a contradiction since for

$$y(s) = (-a(s)\frac{d}{ds}\gamma(\frac{1}{\widehat{q}(s)}\frac{dk}{ds}(s)) - -a(s)\frac{d}{ds}\gamma(\frac{1}{\widehat{q}(s)}\frac{dK}{ds}(s)))$$

we know that $\tau(z,y) \geq 0$ (the T-accretiveness in $L^\infty(\widetilde{\Omega})$ of A), for

$$y(s) = k(s) - K(s)$$

we get that

$$\tau(z,y) = \left\| [k-K]_+ \right\|_{L^\infty(\widetilde{\Omega})} > 0$$

and for

$$y(s) = \gamma(\frac{1}{\widehat{q}(s)}\frac{dk}{ds}(s)) - \gamma(\frac{1}{\widehat{q}(s)}\frac{dK}{ds}(s))$$

we have that

$$\tau(z,y) = 0. \tag{16}$$

The proof of (16) uses the definition of the set $J(z,\varepsilon)$, the fact that $\frac{dk}{ds}, \frac{dK}{ds} \in C^1(\delta, |\Omega|)$ for any $\delta > 0$, and that then there exists $s_0 \in (0, |\Omega|)$ such that

$$\max_{s\in[0,|\Omega|]} [k(s) - K(s)]_+ = k(s_0) - K(s_0),$$

since obviously $s_0 > 0$ and $s_0 < |\Omega|$. Then

$$\frac{dk}{ds}(s_0) = \frac{dK}{ds}(s_0)$$

and hence

$$\lim_{\varepsilon\searrow 0} ess \sup\{\left[\gamma(\frac{1}{\widehat{q}(s)}\frac{dk}{ds}(s)) - \gamma(\frac{1}{\widehat{q}(s)}\frac{dK}{ds}(s))\right] : s \in J(z,\varepsilon)\} = 0.$$

5^{rd} step. The inequality (15) is stable by passing to the limit in a regularizing process on q and so we can assume merely that $q(x)$ satisfies (2). Moreover, by using the Hardy-Littelwood-Polya property (see e.g. [11]) we know that (15) implies the comparison

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq \left\|\overline{U}^\sharp_\lambda\right\|_{L^\infty(\Omega^*)}.$$

Finally, since

$$\left\|\overline{U}^\sharp_\lambda\right\|_{L^\infty(\Omega^*)} \leq \left\|\overline{U}_\lambda\right\|_{L^\infty(\Omega^*)} \leq \left(\frac{q_0}{\lambda}\right)^{\frac{1}{1-\alpha}} \left\|\overline{U}_{\lambda^*}\right\|_{L^\infty(\Omega^*)}$$

we get the estimate (8). The proof of the compactness of the support of the constructed solutions follows the same arguments than the proof of Theorem 5.1 since the key ingredient is to have estimate (8).■

Remark 2 *It is illustrative to compare the above bifurcation results with the ones which are available in the literature for the opposite sign of coefficient $q(x)$, i.e. when $q(x) < -q_0 < 0$. In this case there is again a phenomenon of bifurcation from the infinity near the first eigenvalue of the Laplacian λ_1 but now the existence of positive solutions requires the condition $\lambda < \lambda_1$: see, e.g. [28]*

Remark 3 *The estimate (8) can be also obtained for some solutions of the associated periodic problem*

$$(P)\begin{cases} u_t - \triangle u + q(x)|u|^{\alpha-1}u = \lambda u & \text{in } Q := \Omega\times\mathbb{R}, \\ u(x,t) = 0 & \text{on } \Sigma := \partial\Omega\times\mathbb{R}, \\ u(x,t+T) = u(x,t) & \text{in } Q. \end{cases}$$

The techniques to get a similar treatment of the radial case (and spatially compact support solutions) were given in [2] for a family of related problems. The symmetrization of time-periodic problems was already *studied in [25] for some other eigenvalue problems but the extension to the above problem can be obtained easily. We point out that in [22] it was shown the reverse inequality*

$$\|u_\lambda(t,.)\|_{L^\infty(\Omega)} \geq \frac{C}{\lambda^{\frac{1}{1-\alpha}}} \tag{17}$$

for some $C > 0$ (see their Theorem 3.4). In the stationary case we also can prove a reverse type inequality as (17) at least for the one-dimensional and radially symmetric formulations.

Remark 4 *Theorem 1 can be extended to a quasilinear version of problem $P(q, \alpha, \lambda)$ by replacing the Laplacian operator by the p-Laplacian diffusion*

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \ p > 1.$$

The question is much more delicate (and in fact it is an interesting open problem) when we replace the Laplacian operator by an anisotropic diffusion of the type

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}), \ p_i > 1.$$

The basic comparison result dealing the p-Laplacian can be already found in the references indicated in the proof of Theorem 1. The case of anisotropic diffusion was studied recently in [1]. The associated eigenvalue problem has now a richer structure since the associated eigenvalue problem is not so easily defined as for the p-Laplacian case (see references in [6]). Notice that the arguments of the Proposition 1 do not apply (at least directly) to the case of anisotropic diffusion.

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