Poisson summation formulae and the wave equation with a finitely supported measure as initial velocity.

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Abstract

New Poisson summation formulae have been recently discovered by Nir Lev and Alexander Olevskii since 2013. But some other examples were concealed in an old paper by Andrew Guinand dating from 1959. This was unveiled in a recent paper (2016) by the second author. In the present contribution a third approach is proposed and Guinand's work will be related to the wave equation on the three dimensional torus with a finitely supported measure as initial velocity. Furthermore this new approach shows that, in fact, Guinands solution belongs to a general family of initial velocities giving rise to the corresponding family of crystalline measures, such as it is proved in the last theorem of this paper.

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1 Definition of Poisson's summation formulae

Definition 1.1 *The Fourier transform* $\mathcal{F}(f) = \widehat{f}$ *of a function* f *is defined by* $\widehat{f}(y) = \int_{\mathbb{R}^n} \exp(-2\pi\, ix \cdot y) f(x)\, dx$.

Definition 1.2 *A set of points* $\Lambda \subset \mathbb{R}^n$ *is locally finite if, for every compact set* B, $\Lambda \cap B$ *is finite.*

A locally finite set $E \subset \mathbb{R}^n$ can be ordered as a sequence of points tending to infinity. A set of points $\Lambda \subset \mathbb{R}^n$ is uniformly discrete if

$$\inf_{\{\lambda,\,\lambda'\in\Lambda,\,\lambda'\neq\lambda\}} |\lambda' - \lambda| = \beta > 0 \tag{1}$$

Definition 1.3 *A crystalline measure is an atomic measure μ on $\mathbb{R}^n$ which satisfies the three following conditions:*

- (a) *μ is supported by a locally finite set*
- (b) *μ is a tempered distribution*
- (c) *the distributional Fourier transform $\widehat{\mu}$ of μ is also an atomic measure supported by a locally finite set.*

Let Λ be the support of a crystalline measure μ and let S be its spectrum, i.e. the support of $\widehat{\mu}$. We then have

$$\mu = \sum_{\lambda\in\Lambda} a(\lambda)\delta_\lambda, \ \ \widehat{\mu} = \sum_{y\in S} b(y)\delta_y. \tag{3}$$

It yields the following generalized Poisson's summation formula:

$$\sum_{\lambda\in\Lambda} a(\lambda)\widehat{f}(\lambda) = \sum_{y\in S} b(y)f(y) \tag{4}$$

if $f \in \mathcal{S}(\mathbb{R}^n)$.

A well known example is given by the standard Poisson's summation formula where Λ is a lattice. A lattice $\Gamma \subset \mathbb{R}^n$ is defined by $\Gamma = A\,\mathbb{Z}^n$ where $A \in GL(n,\mathbb{R})$. A Dirac comb is a sum $\mu = \sum_{\gamma\in\Gamma}\delta_\gamma$ of Dirac masses δ_γ on a lattice Γ. The Fourier transform of the Dirac comb on a lattice Γ is (up to a constant factor) the Dirac comb on the dual lattice Γ^*. This is the *standard Poisson's summation formula* which plays a seminal role in X-ray crystallography and molecular biology. Other Poisson's summation formulae directly follow from the standard one.

Definition 1.4 *Let σ_j be a Dirac comb supported by a coset $x_j + \Gamma_j$ of a lattice $\Gamma_j \subset \mathbb{R}^n$, $1 \le j \le N$. Let g_j be a finite trigonometric sum and $\mu_j = g_j\,\sigma_j$. Then $\mu = \mu_1 + \cdots + \mu_N$ will be called a generalized Dirac comb.*

The Fourier transform of a generalized Dirac comb is a generalized Dirac comb.

Definition 1.5 *A crystalline measure* μ *is an exotic crystalline measure if its support* Λ *is not contained in a finite union* $\bigcup_1^N (a_j + \Gamma_j)$ *of co-sets of lattices.*

Our goal is the construction of exotic crystalline measures.

2 Quasi-crystals and crystalline measures

In this section quasi-crystals are defined as model sets [8]. If Λ is a quasi-crystal which is not a lattice, then $\mu = \sum_{\lambda \in \Lambda} \delta_\lambda$ is not a crystalline measure: the Fourier transform of μ is not even a measure. Using the cut and projection construction of quasi-crystals one can fix this issue and find some weights $c(\lambda) \in [0, 1]$ such that the Fourier transform $\widehat{\mu}$ of the measure $\mu = \sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda$ is an atomic measure. However in this construction the spectrum of μ is dense in $\mathbb{R}^n$. Therefore μ is not a crystalline measure. This led to the following conjecture:

Conjecture 2.1 *The support an exotic crystalline measure cannot be contained in a quasi-crystal.*

This conjecture has been finally proved by Nir Lev and Alexander Olevskii in [6].

Theorem 2.1 *Let* μ *be a crystalline measure on* $\mathbb{R}^n$*. Assume that the set* $\Lambda - \Lambda$ *is uniformly discrete, and that* S *is a discrete closed set. Then* μ *is a generalized Dirac comb.*

This implies that quasi-crystals are useless in the construction of crystalline measures. However Nir Lev and Alexander Olevskii used a ladder of quasi-crystals to circumvent the problem. They proved the existence of exotic crystalline measures in [5]. In their construction the support Λ of their crystalline measure μ is contained in the union $\bigcup_0^\infty \Lambda_j$ of an increasing sequence Λ_j of quasi-crystals.

Dirac combs are "isolated points" in the collection of crystalline measures. Nir Lev and Alexander Olevskii proved the following theorem

Theorem 2.2 *In dimension* 1*, if both the support* $\Lambda \subset \mathbb{R}$ *and the spectrum* $S \subset \mathbb{R}$ *of a measure* μ *are uniformly discrete, then* μ *is a generalized Dirac comb.*

Theorem 4.2 holds in dimension $n \geq 2$ if μ is non negative. The general case (uniformly discrete support and spectrum, μ signed measure) is open.

Conjecture 2.2 *A non negative crystalline measure is a generalized Dirac comb.*

3 Guinand's distribution

Let us begin with Guinand's contruction in [1]. By Legendre's theorem, an integer $n \geq 0$ can be written as a sum of three squares (0^2 being admitted) if and only if n is not of the form $4^j(8k+7)$. For instance 0, 1, 2, 3, 4, 5, 6 are sums of three squares but 7 is not. Let $r_3(n)$ be the number of decompositions of the integer $n \geq 1$ into a sum of three squares (with $r_3(n) = 0$ if n is not a sum of three squares). More precisely $r_3(n)$ is the number of points $k \in \mathbb{Z}^3$ such that $|k|^2 = n$. We have $r_3(4n) = r_3(n), \forall n \in \mathbb{N}$, $r_3(0) = 1, r_3(1) = 6, r_3(2) = 12, \ldots$. Then $r_3(2^j) = 6$ if j is even and 12 if j is odd. The behavior of $r_3(n)$ as $n \to \infty$ is erratic. The mean behavior is more regular since

$$\sum_{0 \leq n \leq x} r_3(n) = \frac{4}{3}\pi x^{3/2} + O(x^{3/4}). \tag{5}$$

This is equivalent to

$$\sum_{0 \leq n \leq x} r_3(n) n^{-1/2} = 2\pi x + O(x^{1/4}). \tag{6}$$

Let B_R be the ball centered at 0 with radius R. Then (5) amounts to

$$\#(\mathbb{Z}^3 \cap B_R) = \frac{4}{3}\pi R^3 + O(R^{3/2}).$$

This estimate of the error term is not optimal and 3/2 can be reduced to 21/16 as D. R. Heath-Brown proved in [2]. Guinand began his seminal work [1] with a simple lemma

Lemma 3.1 *For $a > 0$ we have*

$$1 + \sum_{1}^{\infty} r_3(n) \exp(-\pi n a) =$$

(7) $$a^{-3/2} + a^{-3/2} \sum_{1}^{\infty} r_3(n) \exp(-\pi n/a).$$

The simplest proof consists in writing

$$1 + \sum_{1}^{\infty} r_3(n) \exp(-\pi n a) = \sum_{k \in \mathbb{Z}^3} \exp(-\pi a |k|^2)$$

and applying the standard Poisson's formula to the RHS. Let $f_a(x) = x \exp(-\pi a x^2)$, $x \in \mathbb{R},\ a > 0$. Then $f_a(x)$ is odd and its Fourier transform is

$$\widehat{f_a}(y) = -i a^{-3/2} y \exp(-\pi y^2/a).$$

Now (7) can be written

$$\frac{df_a}{dx}(0) + \sum_{1}^{\infty} r_3(n) n^{-1/2} f_a(\sqrt{n}) =$$

(8) $$i \frac{d\widehat{f_a}}{dx}(0) + i \sum_{1}^{\infty} r_3(n) n^{-1/2} \widehat{f_a}(\sqrt{n}).$$

Guinand introduced the odd distribution

(9) $$\sigma = -2 \frac{d}{dx} \delta_0 + \sum_{1}^{\infty} r_3(n) n^{-1/2} (\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})$$

which will be named Guinand's distribution. We have by (6) $\sum_0^N r_3(n) n^{-1/2} = 2\pi N + O(N^{1/4})$ which implies that σ is a tempered distribution. Guinand proved the following

Theorem 3.1 *The distributional Fourier transform of σ is $-i\sigma$.*

The mean behavior at infinity of the Guinand's distribution follows from Theorem 3.1.

Corollary 3.1 *The Guinand's distribution is the sum of a linear trend* $4\pi x$ *and a fluctuation which is an almost periodic distribution. More precisely we have*

$$\sigma(x) = 4\pi x + 2\sum_{1}^{\infty} r_3(n) n^{-1/2} \sin(2\pi\sqrt{n}x). \tag{10}$$

Corollary 3.2

$$\sum_{1}^{\infty} r_3(n) n^{-1/2} \sin(2\pi\sqrt{n}\,x) = 0$$

if $x \neq \sqrt{m}$, $m \in \mathbb{N}$,

$$\sum_{1}^{\infty} r_3(n) n^{-1/2} \sin(2\pi\sqrt{n}\,x) = +\infty$$

if $x = \sqrt{m}$, $m \in \mathbb{N} \setminus \{0\}$.

An Abel summation is needed to sum this divergent series.

We return to Theorem 3.1. We need to prove $\langle \sigma, \widehat{\phi} \rangle = -i\langle \sigma, \phi \rangle$ for every test function ϕ. But (8) can be rewritten as $\langle \sigma, f_a \rangle = i\langle \sigma, \widehat{f_a} \rangle$ or $\langle \sigma, f_a \rangle = i\langle \widehat{\sigma}, f_a \rangle$. The collection of odd functions f_a, $a > 0$, is total in the subspace of odd functions of the Schwartz class. For even functions ϕ the identity $\langle \sigma, \widehat{\phi} \rangle = -i\langle \sigma, \phi \rangle$ is trivial since σ is odd and $\langle \sigma, \widehat{\phi} \rangle = -i\langle \sigma, \phi \rangle = 0$. Theorem 3.1 is proved. This is still a copy of Guinand's paper.

As in [7] we now move one small step beyond Guinand's work. Let $\alpha \in (0, 1)$ and set

$$\tau_\alpha(x) = (\alpha^2 + \frac{1}{\alpha})\,\sigma(x) - \alpha\sigma(\alpha x) - \sigma(x/\alpha). \tag{11}$$

Then the derivative of the Dirac mass at 0 disappears from this linear combination. On the Fourier transform side

$$\widehat{\tau}_\alpha(y) = (\alpha^2 + \frac{1}{\alpha})\,\widehat{\sigma}(y) - \widehat{\sigma}(y/\alpha) - \alpha\widehat{\sigma}(\alpha y) = -i\tau_\alpha(y).$$

Fix $\alpha = 1/2$ in the preceding construction, let $\tau = \tau_{1/2}$ and define $\chi(n) = -1/2$ if $n \in \mathbb{N} \setminus 4\,\mathbb{N}$, $\chi(n) = 4$ if $n \in 4\,\mathbb{N} \setminus 16\,\mathbb{N}$, and $\chi(n) = 0$ if $n \in 16\,\mathbb{N}$. Then we have

Theorem 3.2 *The Fourier transform of the measure*

$$\tau = \sum_1^\infty \chi(n) r_3(n) n^{-1/2} (\delta_{\sqrt{n}/2} - \delta_{-\sqrt{n}/2}) \tag{12}$$

is $-i\tau$.

A more natural proof of Theorem 3.2 will be given in Section 4. A naive corollary is

Corollary 3.3

$$\sum_1^\infty \chi(n) r_3(n) n^{-1/2} \sin(\frac{\pi}{2}\sqrt{n}\, x) = 0$$

if $x \neq \sqrt{m}$, $m \in \mathbb{N}$, *or* $x = 4\sqrt{m}$, $m \in \mathbb{N}$, *while*

$$\sum_1^\infty \chi(n) r_3(n) n^{-1/2} \sin(\frac{\pi}{2}\sqrt{n}\, x) = \infty$$

if $x = \sqrt{m}$, $m \in \mathbb{N} \setminus 16\mathbb{N}$.

Here again an Abel summation is needed to sum this divergent series. By (6)

$$\sum_1^N r_3(n) n^{-1/2} = 2\pi N + O(N^{1/4}),\ N \to \infty,$$

while $\sum_1^\infty \chi(n) r_3(n) = 0$. If χ was erased from (12) τ would no longer be a crystalline measure. The cancellations provided by χ are playing a key role. The measure τ is not an almost periodic measure. Indeed $|\tau|([x, x+1]) \to \infty$, $x \to \infty$. It is however an almost periodic distribution. If μ is a crystalline measure and if $\widehat{\mu} = \lambda\mu$ then $\lambda \in \{1, -1, i, -i\}$. Conversely for each of these four eigenvalues there exists a crystalline measure μ such that $\widehat{\mu} = \lambda\mu$. This will be proved in a forthcoming paper.

4 Guinand's distribution and the wave equation

The Guinand's distribution $\sigma = -2\frac{d}{dt}\delta_0 + \sum_1^\infty r_3(n) n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})$ can also be written as $\sigma = 4\pi t + 2\sum_1^\infty r_3(n) n^{-1/2} \sin(2\pi\sqrt{n}t)$ belong to $\mathcal{S}'(\mathbb{R})$, and this equality is equivalent to Guinand's theorem. As it will be proved below this becomes obvious if one translates it into the language of the wave equation. The proof relies on well known properties of the wave equation on the three dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ which are summarized in the following lemma:

Lemma 4.1 *Let $E = \mathcal{D}'(\mathbb{T}^3)$ denotes the space of Schwartz distributions on $\mathbb{T}^3$. Then for every $u_1 \in E$ there exists a unique solution $u(x,t) \in \mathcal{C}^\infty([0,\infty), E)$ of the Cauchy problem*

(i) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta\, u(x,t)$

(ii) $u(x,0) = 0$, $\frac{\partial}{\partial t} u(x,0) = u_1$.

Moreover $t \mapsto u(x,t)$ extended to $\mathbb{R}$ as an odd function of t belongs to $\mathcal{C}^\infty(\mathbb{R}, E)$

Let $u_1(x) = \sum_{k\in\mathbb{Z}^3} \alpha(k) \exp(2\pi i k\cdot x)$ be the Fourier series expansion of u_1. Then

$$u(x,t) = \alpha(0)t + \frac{1}{2\pi} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \alpha(k) \frac{\sin(2\pi t|k|)}{|k|} \exp(2\pi i k\cdot x). \tag{13}$$

A similar result holds for the wave equation on $\mathbb{R}^3$ where E is replaced by the Schwartz space $\mathcal{S}'$ of tempered distributions on $\mathbb{R}^3$. If we are given a tempered distribution u_1 on $\mathbb{R}^3$ there exists a unique solution $u(x,t)$ of the wave equation $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta\, u(x,t)$ such that $u(x,0) = 0$, $\frac{\partial}{\partial t} u(x,0) = u_1$. It is given by $\widehat{u}(\xi,t) = \frac{\sin(t|\xi|)}{|\xi|}\widehat{u}_1(\xi)$.

Corollary 4.1 *Let $w(x,t)$ be defined on $\mathbb{T}^3 \times \mathbb{R}$ by*

$$w(x,t) = t + \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{\sin(2\pi t|k|)}{2\pi|k|} \exp(2\pi i k\cdot x). \tag{14}$$

Then $w(x,t)$ is the solution to the following Cauchy problem for the wave equation on $\mathbb{T}^3 \times \mathbb{R}$

(i) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta\, u(x,t)$

(ii) $u(x,0) = 0$, $\frac{\partial}{\partial t} u(x,0) = \delta_0(x)$.

Moreover $w(0,t)$ is the RHS of (13). But $w(x,t)$ can also be computed by periodizing the solution of the same Cauchy problem on $\mathbb{R}^3 \times \mathbb{R}$. This scheme is detailed now.

Lemma 4.2 *Let $d\sigma_t$, $t \in \mathbb{R}$, be the normalized surface measure on the sphere $B_t \subset \mathbb{R}^3$ centered at 0 with radius $|t|$ (the total mass of $d\sigma_t$ is 1). Then $v(x,t) = t\, d\sigma_t(x)$ belongs to $\mathcal{C}^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^3))$ and is the solution of the Cauchy problem*

(*i*) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta\, u(x,t)$

(*ii*) $u(x,0) = 0$, $\frac{\partial}{\partial t} u(x,0) = \delta_0(x)$.

Corollary 4.2 *Then*

$$w(x,t) = \sum_{k \in \mathbb{Z}^3} t\, d\sigma_t(x-k) \tag{15}$$

is the solution of the following Cauchy problem for the wave equation on the three dimensional torus:

(*a*) $w(x,0) = 0$

(*b*) $\frac{\partial}{\partial t} w(x,0) = \delta_0(x)$.

The two expansions of $w(x,t)$ given by (14) and (15) are equal and this will imply Guinand's theorem.

Lemma 4.3 *With the preceding notations we have*

$$w(x,t) = \sum_{k \in \mathbb{Z}^3} t\, d\sigma_t(x-k) = t + \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\sin(2\pi t|k|)}{2\pi|k|} \exp(2\pi i k \cdot x). \tag{16}$$

This identity holds in $\mathcal{C}^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^3))$. It can be proved directly. If μ is any compactly supported Borel measure on $\mathbb{R}^3$, the standard Poisson's summation formula yields

$$\sum_{k \in \mathbb{Z}^3} d\mu(x-k) = \sum_{k \in \mathbb{Z}^3} \widehat{\mu}(k) \exp(2\pi i k \cdot x) \tag{17}$$

which implies (16) immediately when $\mu = d\sigma_t$. The detour by the wave equation was only aimed at showing that Guinand's distribution is a natural mathematical object.

Let us compute the trace on $x = x_0$ of the LHS and RHS of (16) as a function of t. This trace is defined as follows.

Definition 4.1 *A distribution $u(x,t) \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ defines a continuous mapping from $\mathbb{R}^3$ to $\mathcal{S}'(\mathbb{R})$ if $< u(x,\cdot), \phi(\cdot) >$ is a continuous function of x for every test function $\phi \in \mathcal{S}(\mathbb{R})$.*

The RHS of (16) fulfills this requirement since $\widehat{\phi}(|k|)$ is rapidly decreasing for $\phi \in \mathcal{S}(\mathbb{R})$. Therefore the trace $w(x_0, t)$ exists for every $x_0 \in \mathbb{R}^3$ and belongs to $\mathcal{S}'(\mathbb{R})$. For computing the trace of the LHS of (16) one uses the following observation:

Lemma 4.4 *For every $x_0 \in \mathbb{R}^3 \setminus \{0\}$, the trace on $x = x_0$ of the tempered distribution $t\sigma_t(\cdot) \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ is $\frac{1}{4\pi|x_0|}(\delta_{|x_0|} - \delta_{-|x_0|})$.*

For checking this fact we observe that $t\sigma_t(\cdot)$ is odd in t. Lemma 4.4 implies the following

Lemma 4.5 *If $x_0 \notin \mathbb{Z}^3$ the trace of $\sum_{k\in\mathbb{Z}^3} t\, d\sigma_t(x-k)$ is $\sum_{k\in\mathbb{Z}^3} \frac{1}{|x_0-k|}(\delta_{|x_0-k|} - \delta_{-|x_0-k|})$.*

We can conclude:

Proposition 4.1 *Let $x_0 \notin \mathbb{Z}^3$. Then we have*

(18)
$$\sum_{k\in\mathbb{Z}^3} \frac{1}{|x_0-k|}(\delta_{|x_0-k|} - \delta_{-|x_0-k|}) = 4\pi t + 2 \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{\sin(2\pi|k|t)}{|k|} \exp(2\pi i k \cdot x_0)$$

and theses two series converge in $\mathcal{S}'(\mathbb{R})$.

This identity does not make sense if $x_0 = 0$ which is needed for recovering Theorem 3.1. The divergence which occurs is responsible for the derivative of the Dirac mass in the LHS of (13). To settle this problem it suffices to observe that the distribution $\sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{\sin(2\pi|k|t)}{|k|} \exp(2\pi i k\cdot x)$ is continuous on $\mathbb{R}^3$. We then compute $w(0,t)$ in (16) as $\lim_{x\to 0,\, x\neq 0} w(x,t)$. Then $\frac{1}{|x_0|}(\delta_{|x_0|} - \delta_{-|x_0|}) \to -2\frac{d}{dt}\delta_0$ as $x_0 \to 0$ which yields a second proof of Theorem 3.1.

Theorem 3.2 is a particular case of a more general fact. The notations are the same as above.

Theorem 4.1 *Let ν a real, finitely supported measure on $\mathbb{T}^3$ such that*

(*a*) 0 *does not belong to the support of ν*

(b) $\int_{\mathbb{T}^3} d\,\nu = 0$.

Let $u : \mathbb{T}^3 \times \mathbb{R} \mapsto \mathbb{R}$ *be the solution of the Cauchy problem*

(i) $\frac{\partial^2}{\partial t^2}\, u(x,t) = \Delta\, u(x,t)$

(ii) $u(x,0) = 0,\ \frac{\partial}{\partial t}\, u(x,0) = \nu.$

Then $t \mapsto u(0,t)$ *is a crystalline measure.*

The proof is identical to the one given above. Theorem 3.2 is now a direct corollary. It suffices to define ν by the following four conditions: ν is supported by $\{k/4,\ k \in \mathbb{Z}^3\}$, ν does not charge $\mathbb{Z}$, the mass of ν on each $k+1/2$ is $1/2$, and the charge of ν on each $k/2+1/4$ is $-1/16$.

The lattice $\mathbb{Z}^3$ is now replaced by an arbitrary lattice $\Gamma \subset \mathbb{R}^3$ and the proof of Theorem 4.1 yields the following result:

Theorem 4.2 *Let* $\Gamma \subset \mathbb{R}^3$ *be a lattice. Let* ν *be a finitely supported measure on* $V = \mathbb{R}^3/\Gamma$ *such that* $\int_V d\,\nu = 0$. *Let us assume that* 0 *does not belong to the support of* ν. *Let* $u : V \times \mathbb{R} \mapsto \mathbb{R}$ *be the solution of the Cauchy problem*

(i) $\frac{\partial^2}{\partial t^2}\, u(x,t) = \Delta\, u(x,t)$

(ii) $u(x,0) = 0,\ \frac{\partial}{\partial t}\, u(x,0) = \nu.$

Then $t \mapsto u(0,t)$ *is a crystalline measure.*

Indeed

$$u(x,t) = \sum_{\gamma^* \in \Gamma^*} \widehat{\nu}(\gamma^*) \frac{\sin(2\pi t|\gamma^*|)}{2\pi|\gamma^*|} \exp(2\pi i x \cdot \gamma^*) \tag{19}$$

and we also have as above

$$u(x,t) = \sum_{\gamma^* \in \Gamma^*} t\, d\sigma_t * \nu(x-\gamma^*). \tag{20}$$

By (20) $u(0,t)$ is an atomic measure and by (19) $u(0,t)$ is the Fourier transform of the atomic measure

$$\mu = \sum_{\gamma^* \in \Gamma^*} \frac{i\,\widehat{\nu}(\gamma^*)}{4\pi|\gamma^*|} (\delta_{|\gamma^*|} - \delta_{-|\gamma^*|}). \tag{21}$$

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