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On the differentiability of very weak solutions with right-hand side data integrable with respect to the distance to the boundary

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Abstract

We study the differentiability of very weak solutions $v \in L^1(\Omega)$ of $(v, L^\star\varphi)_0 = (f, \varphi)_0$ for all $\varphi \in C^2(\overline{\Omega})$ vanishing at the boundary whenever f is in $L^1(\Omega, \delta)$, with $\delta = \operatorname{dist}(x, \partial\Omega)$, and L^* is a linear second order elliptic operator with variable coefficients. We show that our results are optimal. We use symmetrization techniques to derive the regularity in Lorentz spaces or to consider the radial solution associated to the increasing radial rearrangement function $\widetilde{f}$ of f.

Keywords: Very weak solutions; Distance to the boundary; Regularity; Linear PDE; Monotone rearrangement; Lorentz spaces

1. Introduction

The origin of this paper starts with an originally unpublished manuscript by H. Brezis (personal communication of him to the first author [4]), later mostly in the paper by Brezis et al. [5] (see also the mention made in [17]). In his note, when f is given in $L^1(\Omega, \operatorname{dist}(x, \partial\Omega))$

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(Ω bounded smooth open set of $\mathbb{R}^N$), H. Brezis shows the existence and uniqueness of a very weak solution $v \in L^1(\Omega)$ of

$$GD(\Omega) = \begin{cases} -\int_\Omega v\Delta\varphi\,dx = \int_\Omega f\varphi\,dx, & \forall\varphi \in V_1(\Omega), \\ \text{with } V_1(\Omega) = \{\varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega\}, \end{cases}$$

and also that

$$|v|_{L^1(\Omega)} \leqslant c|f|_{L^1(\Omega,\mathrm{dist}(x,\partial\Omega))}.$$

Therefore, the question of the integrability of the generalized derivative $\partial_i v = \frac{\partial v}{\partial x_i}$ arises in a natural way and was raised already in the note by H. Brezis.

To give some answer to the above question, we shall note $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ and introduce the following spaces

$$L^1(\Omega, \delta^\alpha) = \left\{ f : \Omega \to \mathbb{R} \text{ Lebesgue measurable: } \int_\Omega |f(x)|\delta(x)^\alpha\,dx \text{ is finite} \right\},$$

$$0 \leqslant \alpha \leqslant 1, \quad L^1(\Omega, \delta^1) = L^1(\Omega, \delta), \quad L^1(\Omega, \delta^0) = L^1(\Omega),$$

$$L^1(\Omega, \delta|\mathrm{Ln}\,\delta|) = \left\{ f : \Omega \to \mathbb{R} \text{ Lebesgue measurable such that } \int_\Omega |f|(x)\delta(x)|\mathrm{Ln}\,\delta(x)|\,dx < +\infty \right\}.$$

One has, for $\alpha \in [0, 1[$

$$L^1(\Omega, \delta^\alpha) \subsetneqq L^1(\Omega, \delta|\mathrm{Ln}\,\delta|) \subsetneqq L^1(\Omega, \delta).$$

One of our results contains in particular the following statements:

(i) *The very weak solution* $v \in W_0^{1,q}(\Omega)$ *for some* $q > 1$ *if and only if* $f \in L^1(\Omega, \delta^\alpha)$ *for some* $\alpha \in [0, 1[$, *for nonnegative* f.
(ii) *If* $f \in L^1(\Omega, \delta^\alpha)$, $0 \leqslant \alpha < 1$ *then* $|\nabla v|$ *belongs to the Lorentz space* $L^{\frac{N}{N-1+\alpha},\infty}(\Omega)$.

The above result contains the result given in [12] since $L^p(\Omega, \delta) \subsetneqq L^1(\Omega, \delta^{\frac{1}{p}})$ for $p > 1$. We also improve the result of Cabré and Martel [6], by showing that if f is only in $L^1(\Omega, \delta)$ then the function v is in $L^{\frac{N}{N-1},\infty}(\Omega)$. Moreover, we can show that $|\nabla v| \in L^q(\Omega, \delta)$ for some $q > 1$, in particular $v \in W^{1,q}_{loc}(\Omega)$.

As a matter of fact, all our results in the first four sections are valid when we replace the Laplacian operator by a linear elliptic second order operator L with variable coefficients.

In Section 5, we consider the case of $L^* = -\Delta$ and Ω being the unit ball. Our aim is to study if we may have the $W^{1,1}$-regularity whenever

$$f \in L^1(\Omega, \delta) - L^1(\Omega, \delta^{1-}), \quad L^1(\Omega, \delta^{1-}) = \bigcup_{0\leqslant\alpha<1} L^1(\Omega, \delta^\alpha).$$

We show:

(1) If $f \in L^1(\Omega, \delta|\operatorname{Ln}\delta|)$ then the very weak solution $v \in L^1(\Omega)$ is in $W_0^{1,1}(\Omega)$.
(2) If we consider $\widetilde{f}$ the increasing radial rearrangement function (see its definition in Section 2) of $f \in L^1(\Omega, \delta)$, $f \geqslant 0$ then $\widetilde{f} \in L^1(\Omega, \delta)$. Moreover, the unique very weak solution $\omega \in L^1(\Omega)$ of the generalized Dirichlet problem $GD(\Omega)$,

$$-\int_\Omega \omega(x)\Delta\varphi(x)\,dx = \int_\Omega \widetilde{f}\varphi(x)\,dx, \quad \forall\varphi \in V_1(\Omega),$$

is radial but *decreasing* belonging to $W_0^{1,1}(\Omega)$. Moreover, there exists a constant $c(\Omega) > 0$

$$|\nabla\omega|_{L^1(\Omega)} \leqslant c(\Omega) \cdot |f|_{L^1(\Omega,\delta)}.$$

Under our assumptions on Ω,

$$c(\Omega) = \frac{1}{N\alpha_N^{1+\frac{1}{N}}}.$$

We shall restate the necessary and sufficient conditions to ensure that $\omega \in W^{1,q}(\Omega)$ for $q > 1$ and we shall show that

$$\omega \in W_0^{1,q}(\Omega) \quad \text{if and only if} \quad \int_0^{|\Omega|}\left(\int_\sigma^{|\Omega|} f_*(t)\,dt\right)^q d\sigma \text{ is finite.}$$

We also remark that the usual comparison technique based on the decreasing rearrangement $f_*(t)$ of $f \geqslant 0$, is inefficient in the case where $f \in L^1(\Omega, \delta) \setminus L^1(\Omega)$. Indeed, the function

$$U(x) = c_N \int_{\alpha_N|x|^N}^{|\Omega|} t^{\frac{2}{N}-2}\int_0^t f_*(\sigma)\,d\sigma$$

is in $L^1(\Omega)$ if and only if $f \in L^1(\Omega)$. In any case the pointwise comparison $\underset{\sim}{v} \leqslant U$ and the comparison in mass (see, e.g. the results and references presented in Section 1.3 of [7]) are still true (but they do not give any information on the integrability of v). We end the paper by giving two applications of our differentiability results to two special data f which are in $L^1(\Omega, \delta)$ but not in $L^1(\Omega)$ nor in $L^1(\Omega, \delta^\alpha)$, respectively. The application to the existence, uniqueness and qualitative properties of the very weak solution of some associate semilinear problem will be the object of a separate paper by the authors (Díaz and Rakotoson [8]).

2. Notation – preliminary results

We shall always consider $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, a bounded open set of class $C^{2,1}$. For any measurable set $E \subset \mathbb{R}^N$ we shall denote by $|E|$ its Lebesgue measure.

We shall consider a linear operator L:

$$Lu = -\sum_{i,j=1}^{N} \partial_i\big(a_{ij}(x)\partial_j u\big) + \sum_{i=1}^{N} b^i(x)\partial_i u + c_0(x)u$$

under the same assumptions as in [9], say $a_{ij} \in C^{0,1}(\overline{\Omega})$, $b^i \in C^{0,1}(\overline{\Omega})$, $c_0 \in L^\infty(\Omega)$, $c_0 \geqslant 0$, $\forall \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geqslant \alpha|\xi|^2 \quad \text{for some } \alpha > 0, \qquad c_0(x) - \frac{1}{2}\sum_{i=1}^{N} \partial_i b^i(x) \geqslant 0 \quad \text{a.e. in } \Omega.$$

We shall use the adjoint operator associated to L, that is

$$L^*\varphi = -\sum_{i,j} \partial_j\big(a_{ij}(x)\partial_i\varphi\big) - \sum_{i=1}^{N} \partial_i\big(b^i\varphi\big) + c_0(x)\varphi.$$

Remark 1. The case of unbounded term $c_0(x)$, blowing up on the boundary, will be considered in a subsequent paper by the authors (Díaz and Rakotoson [8]) where in fact the general framework will concern the case of semilinear equations.

We recall that:

– the *decreasing rearrangement of a measurable function u* is given by

$$u_* : \Omega_* = \big]0, |\Omega|\big[\to \mathbb{R}, \quad u_*(s) = \inf\big\{t \in \mathbb{R}\colon |u > t| \leqslant s\big\},$$
$$u_*(0) = \operatorname*{ess\,sup}_{\Omega} u, \qquad u_*\big(|\Omega|\big) = \operatorname*{ess\,inf}_{\Omega} u;$$

– the *decreasing radial rearrangement of the function u* is defined, on the ball $\widetilde{\Omega}$ having the same measure as Ω, by

$$\underset{\sim}{u} : \widetilde{\Omega} \to \mathbb{R}, \quad \underset{\sim}{u}(x) = u_*\big(\alpha_N|x|^N\big);$$

– the *increasing rearrangement of a measurable function u* is given by

$$u^* : \Omega_* \to \mathbb{R}, \quad u^*(s) = u_*\big(|\Omega| - s\big), \quad s \in \big]0, |\Omega|\big[;$$

– the *increasing radial rearrangement of the function u* is defined by

$$\widetilde{u} : \widetilde{\Omega} \to \mathbb{R}, \quad \widetilde{u}(x) = u^*\big(\alpha_N|x|^N\big).$$

We shall use the following Lorentz spaces (see [14,1] for example), for $1 < p < +\infty$, $1 \leqslant q \leqslant +\infty$

$$L^{p,q}(\Omega) = \left\{ v : \Omega \to \mathbb{R} \text{ measurable } |v|^q_{L^{p,q}} = \int_0^{|\Omega|} \left[t^{\frac{1}{p}} |v|_{**}(t) \right]^q \frac{dt}{t} < +\infty \right\},$$

for $q < +\infty$

$$L^{p,\infty}(\Omega) = \left\{ v : \Omega \to \mathbb{R} \text{ measurable } |v|_{L^{p,\infty}} = \sup_{t \leqslant |\Omega|} t^{\frac{1}{p}} |v|_{**}(t) < +\infty \right\},$$

χ_E is the characteristic function of a set $E \subset \Omega$ and $|v|_{**}(t) = \frac{1}{t}\int_0^t |v|_*(s)\,ds$ for $t \in \Omega_* =]0, |\Omega|[$.

We denote by $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. We define the following sets

$$W^1\big(\Omega, |\cdot|_{p,q}\big) = \big\{ v \in W^{1,1}(\Omega)\colon\ |\nabla v| \in L^{p,q}(\Omega) \big\}$$

and

$$W^2\big(\Omega, |\cdot|_{p,q}\big) = \big\{ v \in W^{2,1}(\Omega)\colon\ \partial_{ij} v \in L^{p,q}(\Omega) \text{ for } (i,j) \in \{1,\dots,N\}^2 \big\}.$$

We shall denote by c various constants depending only on the data.

The notation $\approx$ stands for equivalence of nonnegative quantities, that is

$$\Lambda_1 \approx \Lambda_2 \quad \Longleftrightarrow \quad \exists c_1 > 0,\ c_2 > 0 \text{ such that } c_1 \Lambda_2 \leqslant \Lambda_1 \leqslant c_2 \Lambda_2.$$

We first extend the Agmon–Douglis–Nirenberg theorem to Lorentz spaces.

Lemma 1. *Consider L^* the above linear operator. There exists a constant $c(\Omega, L^*) > 0$ such that $\forall g \in L^{N,1}(\Omega)$ there exists a function $\varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H^1_0(\Omega)$ satisfying*

$$L^* \varphi = g,$$

and

$$|\varphi|_{H^1} + \operatorname*{Max}_{i,j} |\partial_{ij}\varphi|_{L^{N,1}} \leqslant c(\Omega, L^*) |g|_{L^{N,1}}.$$

Proof. For $g \in L^2(\Omega)$, we know (see [9]) that there exists a unique function $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$ such that $L^*\varphi = g$. This defines a continuous linear operator A from $L^2(\Omega)$ into $H^2(\Omega)$ by setting $Ag = \varphi$. Let $(i,j) \in \{1,\dots,N\}^2$, $x \in \Omega$, we define

$$T_{ij} g(x) = \partial_{ij} A g(x),$$

T_{ij} is a linear map acting continuously from $L^p(\Omega)$ into $L^p(\Omega)$ for all $p \in [2, +\infty[$ according to Agmon–Douglis–Nirenberg's theorem [9], we derive from Marcinkiewicz's interpolation theorem (see [1]) that it maps continuously

$$L^{N,1}(\Omega) \text{ into } L^{N,1}(\Omega): \quad |T_{ij} g|_{L^{N,1}} \leqslant c(\Omega, L^*)|g|_{L^{N,1}}.$$

This shows the lemma with the fact that

$$|\nabla A g|_{L^2} \leqslant c|g|_{L^2} \leqslant c|g|_{L^{N,1}(\Omega)}. \qquad \square$$

3. General result for $f \in L^1(\Omega, \delta)$

The following existence theorem follows the idea of H. Brezis [4] and the regularity improves the one obtained in [6] for the case $L = \delta$ and in [17] for the case of a general operator L.

Theorem 1. *Let $f \in L^1(\Omega, \delta)$ and $N' = \frac{N}{N-1}$. Then there exists a unique function $v \in L^{N',\infty}(\Omega)$ satisfying*

$$\big(DG_L(\Omega)\big): \quad \int_\Omega v L^* \varphi \, dx = \int_\Omega f \varphi \, dx, \quad \forall \varphi \in W^2\big(\Omega, |\cdot|_{N,1}\big) \cap H_0^1(\Omega).$$

Moreover, there exists a constant $c(\Omega, L) > 0$ such that

$$|v|_{L^{N',\infty}} \leqslant c(\Omega, L)|f|_{L^1(\Omega,\delta)}. \tag{1}$$

Proof. For $k \geqslant 1$, we define the usual truncation

$$T_k(\sigma) = \begin{cases} \sigma & \text{if } |\sigma| \leqslant k, \\ k \operatorname{sign}(\sigma) & \text{otherwise,} \end{cases} \quad \sigma \in \mathbb{R}.$$

We set $f_k = T_k(f) \in L^1(\Omega, \delta) \cap L^\infty(\Omega)$ and $f_k \to f$ in $L^1(\Omega, \delta)$. By standard result there exists a unique function

$$v_k \in W^{2,p}(\Omega) \cap H_0^1(\Omega), \quad \forall p \in [1, +\infty[: \quad L v_k = f_k.$$

Next we want to show that v_k is a Cauchy sequence in $L^{N',\infty}(\Omega)$. For $n \geqslant 1$, $k \geqslant 1$, we set $v^{nk} = v_n - v_k$, $f^{nk} = f_n - f_k$. Then $L v^{nk} = f^{nk}$ which implies that $\forall \varphi \in H^2(\Omega) \cap H_0^1(\Omega)$

$$\int_\Omega v^{nk} L^* \varphi \, dx = \int_\Omega f^{nk} \varphi \, dx. \tag{2}$$

For any E measurable in Ω, there exists a function $\varphi_E \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ such that

$$L^* \varphi_E = \chi_E \operatorname{sign}\big(v^{nk}\big). \tag{3}$$

From Sobolev embedding associated to Lorentz spaces (see [14]), we have

$$|\nabla\varphi_E|_\infty \leqslant c(\Omega)\Big[\operatorname*{Max}_{i,j}|\partial_{ij}\varphi_E|_{L^{N,1}} + |\varphi_E|_{H^1}\Big] \tag{4}$$

and using Lemma 1, we derive that

$$|\nabla\varphi_E|_\infty \leqslant c|\chi_E|_{L^{N,1}} \leqslant c|E|^{\frac{1}{N}}. \tag{5}$$

Since $\forall x \in \Omega$, we have

$$\left|\frac{\varphi_E(x)}{\delta(x)}\right| \leqslant c|\nabla\varphi_E|_\infty, \tag{6}$$

and from relations (5) and (6), we get

$$\frac{|\varphi_E(x)|}{\delta(x)} \leqslant c|E|^{\frac{1}{N}}, \quad \forall x \in \Omega. \tag{7}$$

From relations (2) and (3), we derive

$$\int_E |v_n - v_k|\,dx = \int_\Omega f^{nk}\varphi_E\,dx. \tag{8}$$

From relations (7) and (8), we have

$$\int_E |v_n - v_k|\,dx \leqslant c|E|^{\frac{1}{N}} \int_\Omega |f_n - f_k|(x)\delta(x)\,dx \tag{9}$$

for all E measurable sets in Ω.

Using the Hardy–Littlewood inequality (see [14,1]), we have

$$\sup_{t\leqslant|\Omega|}\big[t^{1-\frac{1}{N}}|v_n - v_k|_{**}(t)\big] \leqslant c|f_n - f_k|_{L^1(\Omega,\delta)}. \tag{10}$$

This shows the result.

Knowing that $L^{N',\infty}$ is the dual and associate space of $L^{N,1}$ we pass to the limit in relation that

$$\int_\Omega v_k L^*\psi\,dx = \int_\Omega f_k\psi\,dx, \quad \forall\psi \in W^2\big(\Omega, |\cdot|_{N,1}\big) \cap H_0^1(\Omega) \tag{11}$$

as $k \to +\infty$ to derive the result. □

Next, we want to show that the solution is in $W^{1,q}(\Omega,\delta)$ for $\frac{2N}{2N-1} > q$ provided that $c(x) = 0 = b^i(x)$ and $a_{ij} = a_{ji}$.

Lemma 2. *Assume L is the self-adjoint uniformly elliptic operator $L = -\sum_{i,j} \partial_i(a_{i,j}(\cdot)\partial_j)$. Then there exists a function $\varphi_1 \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ and $\lambda_1 > 0$, $\forall p \in]1, +\infty[$ satisfying*

$$\begin{cases} L\varphi_1 = \lambda_1 \varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0. \end{cases}$$

Moreover, there are two constants $c_1 > 0$, $c_2 > 0$ such that

$$c_1 \delta(x) \leqslant \varphi_1(x) \leqslant c_2 \delta(x) \quad \forall x \in \Omega.$$

Proof. The proof of the existence is classical (see [15,5]). The estimate is a consequence of Hopf lemma and can be proved as in the case $L = -\Delta$ (see [12,2]). □

Theorem 2. *Under the same assumptions as for Lemma 2, the unique generalized function v given in Theorem 1 belongs to $W^{1,q}(\Omega, \delta)$ for $1 \leqslant q < \frac{2N}{2N-1}$.*

Proof. Let us show that the sequence $v_k \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ $\forall p \in [1, +\infty[$ solution of $Lv_k = f_k$ is a Cauchy sequence in $W^{1,q}(\Omega, \delta)$.

For $\eta > 0$ (that we shall choose later), we consider

$$\phi_\eta(\sigma) = \int_0^\sigma \frac{dt}{(1+t^2)^{\frac{1+\eta}{2}}}, \quad \sigma \in \mathbb{R},$$

the function $\psi_k = \phi_\eta(v_k)\varphi_1$, with φ_1 the first eigenfunction associated to L. Then,

$$\sum_{i,j} \int_\Omega a_{ij}(x)\partial_i v_k \partial_j \psi_k \, dx = \int_\Omega f_k \psi_k \, dx. \tag{12}$$

Using the coercivity condition on a_{ij}, we have

$$\alpha \int_\Omega \frac{|\nabla v_k|^2}{(1+v_k^2)^{\frac{1+\eta}{2}}} \varphi_1 \, dx + \sum_{i,j} \int_\Omega a_{ij}(x)\partial_j \varphi_1 \partial_i \left(\int_0^{v_k} \phi_\eta(t)\, dt \right) dx \leqslant \int_\Omega f_k \psi_k \, dx. \tag{13}$$

We have

$$\begin{aligned} \sum_{i,j} \int_\Omega a_{ij}(x)\partial_j \varphi_1 \partial_i \left(\int_0^{v_k} \phi_\eta(t)\, dt \right) dx &= \int_\Omega \left[\int_0^{v_k} \phi_\eta(t)\, dt \right] L\varphi_1 \, dx \\ &= \lambda_1 \int_\Omega \varphi_1 \left(\int_0^{v_k} \phi_\eta(t)\, dt \right) dx. \end{aligned} \tag{14}$$

From relations (13) and (14), we derive using Lemma 2

$$\int_\Omega \frac{|\nabla v_k|^2}{(1+v_k^2)^{\frac{1+\eta}{2}}}\delta(x)\,dx \leqslant c|\phi_\eta|_\infty\left[\int_\Omega \varphi_1|v_k|\,dx + \int_\Omega f_k\varphi_1\,dx\right] \leqslant c(\eta)\int_\Omega f_k(x)\delta(x)\,dx. \tag{15}$$

We conclude as in [13] (see also [3] for another proof), using Hölder inequality, with $q \in [1, \frac{2N}{2N-1}[$, we have

$$\int_\Omega |\nabla v_k|^q(x)\delta(x)\,dx \leqslant c|f_k|^{\frac{q}{2}}_{L^1(\Omega,\delta)}\left(1+\int_\Omega |v_k|^m(x)\delta(x)\,dx\right)^{1-\frac{q}{2}} \tag{16}$$

with $m = \frac{(1+\eta)q}{2-q} < \frac{N}{N-1}$ (choice of η).

Since

$$|v_k|_{L^m(\Omega,\delta)} \leqslant c|v_k|_{L^{N',\infty}} \leqslant c|f_k|_{L^1(\Omega,\delta)},$$

we then have

$$\int_\Omega |\nabla v_k|^q(x)\delta(x)\,dx \leqslant c|f_k|^{\frac{q}{2}}_{L^1(\Omega,\delta)}\big(1+|f|_{L^1(\Omega,\delta)}\big)^{1-\frac{q}{2}}. \tag{17}$$

By linearity of the equation

$$\big|\nabla(v_k - v_n)\big|_{L^q(\Omega,\delta)} \leqslant c|f_k - f_n|^{\frac{q}{2}}_{L^1(\Omega,\delta)} \to 0 \quad \text{as } n,k\to+\infty.$$

We conclude that $v \in W^{1,q}(\Omega,\delta)$. □

4. Necessary and sufficient conditions for the $L^q(\Omega)$-integrability of gradient

In this section, we are investigating the integrability of the gradient on the whole set Ω for the general operator L.

We start with the following theorem:

Theorem 3. *Let v be the unique solution of $(DG_L(\Omega))$ given in Theorem* 1.

If $f \in L^1(\Omega,\delta^\alpha)$ for some $\alpha \in [0,1[$, then

$$|\nabla v| \in L^{\frac{N}{N-1+\alpha},\infty}(\Omega).$$

Moreover, there exists $c(\Omega,L) > 0$

$$|\nabla v|_{L^{\frac{N}{N-1+\alpha},\infty}} \leqslant c(\Omega,L)|f|_{L^1(\Omega,\delta^\alpha)}.$$

The proof relies on the following result which is a consequence of Simader's result [16].

Lemma 3. *For $u \in L^1_{loc}(\Omega)$, we consider a measurable vector field $H(u) \in L^\infty(\Omega)^N$.*

For any function $g \in L^p(\Omega)$, $2 \leqslant p < +\infty$ there exists a unique function $\varphi \in W_0^{1,p}(\Omega)$ such that

$$B(\varphi, \psi) \doteq \sum_{i,j} \int_\Omega a_{ij}(x)\partial_i\varphi\partial_j\psi\,dx + \int_\Omega \sum_i b_i(x)\varphi\partial_i\psi\,dx + \int_\Omega c_0\varphi\psi\,dx$$
$$= \int_\Omega g(x)H(u)\cdot\nabla\psi\,dx, \quad \forall\psi \in W_0^{1,p'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Moreover, there exists a constant $c = c(\Omega, L, p) > 0$ (independent of φ) such that

$$|\varphi|_{W_0^{1,p}(\Omega)} \leqslant c\big|H(u)g\big|_{L^p(\Omega)}. \tag{18}$$

Proof. If $p = 2$ it is a consequence of Lax–Milgram theorem. We notice that

$$B(\varphi, \varphi) \geqslant \alpha|\nabla\varphi|_{L^2}^2 + \int_\Omega \left[c_0 - \frac{1}{2}\sum_{i=1}^N \frac{\partial b_i}{\partial x_i}\right]\varphi^2\,dx \geqslant \alpha|\nabla\varphi|_{L^2}^2$$

for $\varphi \in H_0^1(\Omega)$ and then

$$|\nabla\varphi|_{L^2}^2 + |\varphi|_{L^2}^2 \leqslant c\big|H(u)g\big|_{L^2(\Omega)}^2. \tag{19}$$

If $p > 2$, we apply Simader's result to derive the regularity of the above unique function $\varphi \in W_0^{1,p}(\Omega)$.

Moreover, there exists a constant $\gamma(\Omega, p, L) > 0$:

$$|\varphi|_{W^{1,p}} \leqslant \gamma\big(\big|H(u)g\big|_{L^p} + |\varphi|_{L^p}\big). \tag{20}$$

Since

$$|\varphi|_{L^2} \leqslant c\big|H(u)g\big|_{L^2} \leqslant c\big|H(u)g\big|_{L^p}, \tag{21}$$

and

$$|\varphi|_p \leqslant c|\varphi|_{L^2}^\theta|\varphi|_{W_0^{1,p}}^{1-\theta} \quad \text{for some } \theta \in]0,1[, \tag{22}$$

one derives via Young's inequality, relations (20) to (22)

$$|\varphi|_{W^{1,p}} \leqslant c\big|H(u)g\big|_{L^p}. \qquad \square$$

We shall use a corollary of Lemma 3.

Corollary 3.1 of Lemma 3. *Under the same assumptions as for Lemma* 3*, for all* $p \geqslant 2$*, all* $r \in [1, +\infty]$ *if* $g \in L^{p,r}(\Omega)$ *then the unique solution* φ *of*

$$B(\varphi, \psi) = \int_\Omega g H(u) \nabla\psi \, dx, \quad \forall\psi \in W_0^{1,p'}(\Omega), \tag{23}$$

belongs to $W^1(\Omega, |\cdot|_{p,r})$ *and*

$$|\nabla\varphi|_{L^{p,r}(\Omega)} \leqslant c \big|H(u)g\big|_{L^{p,r}(\Omega)}. \tag{24}$$

Moreover, for $p \geqslant N$*, for all* $x \in \Omega$

$$|\varphi(x)| \leqslant c_p \big|H(u)g\big|_{L^{p,1}} \cdot \delta(x)^{1-\frac{N}{p}}. \tag{25}$$

Proof. To deduce the relation (24), we apply the Marcinkiewicz's interpolation theorem (see [1]) with $Tg = |\nabla Ag|$, where the map A is defined as $A(g) = \varphi$ with φ the unique solution of (23). T maps $L^p(\Omega)$ into $L^p(\Omega)$ continuously and then from $L^{p,r}(\Omega)$ into itself. Therefore, we have relation (24) thanks to Lemma 3. While for relation (25), we use the Sobolev embedding $W^1(\Omega, |\cdot|_{p,1}) \subset_{>} C^{0,\beta}(\overline{\Omega})$ with $\beta = 1 - \frac{N}{p}$ if $p > N$ and $W^1(\Omega, |\cdot|_{N,1}) \subset_{>} C(\Omega) \cap L^\infty(\Omega)$ if $p = N$ (see [14]). We combine these results with relation (24) to derive the result. $\square$

Proof of Theorem 3. We shall consider $v_k \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ $\forall p \in [1, +\infty[$ satisfying

$$Lv_k = f_k = T_k(f).$$

We want to show that $(v_k)_{k\geqslant 1}$ is a Cauchy sequence in $W^1(\Omega, |\cdot|_{q_\alpha,\infty})$ with $q_\alpha = \frac{N}{N-1+\alpha}$.

We introduce

$$H(v_k) = \begin{cases} \frac{\nabla v_k}{|\nabla v_k|} & \text{if } \nabla v_k \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any E measurable $\subset \Omega$, we have from Lemma 3 and its corollary a function $\varphi_E \in W^1(\Omega, |\cdot|_{p,1})$ $\forall p \in [2, +\infty[$ such that

$$B(\varphi_E, \psi) = \int_E H(v_k) \cdot \nabla\psi \, dx \quad \forall\psi \in W^{1,p'}(\Omega) \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

Choosing $\psi = v_k$, we have

$$\int_E |\nabla v_k| \, dx = B(\varphi_E, v_k) = \int_\Omega \varphi_E L v_k \, dx = \int_\Omega f_k \varphi_E \, dx. \tag{26}$$

From relation (25), we know that

$$\big|\varphi_E(x)\big| \leqslant c|\chi_E|_{L^{p,1}} \cdot \delta(x)^{1-\frac{N}{p}}. \tag{27}$$

So let us fix $\alpha \in [0, 1[$ and choose p so that

$$\alpha = 1 - \frac{N}{p} \quad \text{that is} \quad \frac{1}{p} = \frac{1-\alpha}{N}. \tag{28}$$

Therefore, from relations (26) and (27) one has

$$\int_E |\nabla v_k| \leqslant c|\chi_E|_{L^{p,1}} \cdot |f_k|_{L^1(\Omega,\delta^\alpha)}. \tag{29}$$

Since $|\chi_E|_{L^{1,p}} \leqslant c_p|E|^{\frac{1}{p}}$, one has from relation (29)

$$\sup_{t \leqslant |\Omega|} \left[t^{1-\frac{1}{p}} |\nabla v_k|_{**}(t)\right] \leqslant c|f_k|_{L^1(\Omega,\delta^\alpha)} \quad \text{with } 1 - \frac{1}{p} = \frac{N-1+\alpha}{N} = \frac{1}{q_\alpha}. \tag{30}$$

By linearity, relation (30) implies that $(v_k)_{k\geqslant 1}$ is a Cauchy sequence in $W^1(\Omega, |\cdot|_{q_\alpha,\infty})$. □

Now, we are able to prove

Theorem 4. *Let v be the unique solution of the generalized Dirichlet problem $(DG_L(\Omega))$, $f \geqslant 0$. Then $v \in W_0^{1,q}(\Omega)$ for some $q > 1$ if and only if there exists $\alpha \in [0, 1[$ such that $f \in L^1(\Omega, \delta^\alpha)$.*

Proof. From Theorem 3, we know that

$$\text{if } f \in L^1\big(\Omega, \delta^\alpha\big) \text{ then } v \in W^1\big(\Omega, |\cdot|_{q_\alpha,\infty}\big) \subset W_0^{1,q}(\Omega) \text{ for all } 1 \leqslant q < q_\alpha.$$

For the converse, we use that $f \geqslant 0$.

If $v \in W_0^{1,q}(\Omega)$, $q > 1$ then we have for all $\varphi \in C_c^\infty(\Omega)$

$$-\int_\Omega v\partial_i\big(a_{ij}(x) \cdot \partial_j\varphi\big)\,dx = \int_\Omega \partial_i v \partial_j \varphi a_{ij}\,dx. \tag{31}$$

We deduce from the equation satisfied by v that $\forall\varphi \in C_c^\infty(\Omega)$, $\varphi \geqslant 0$

$$\int_\Omega f\varphi\,dx = \sum_{i,j}\int_\Omega a_{ij}(x)\partial_i v\partial_j\varphi\,dx + \int_\Omega b^i v\partial_i\varphi\,dx + \int_\Omega c_0 v\varphi\,dx = B(v,\varphi). \tag{32}$$

Using a density argument and Fatou's lemma, the relation (32) implies $\forall\varphi \in W_0^{1,q'}(\Omega)$, $\varphi \geqslant 0$, $\frac{1}{q} + \frac{1}{q'} = 1$

$$\int_\Omega f\varphi\,dx \leqslant B(v,\varphi). \tag{33}$$

We choose $1 > \alpha = 1 - \beta > \frac{1}{q}$, we have $\int_\Omega \delta(x)^{-q'\beta}\,dx < +\infty$, therefore the function

$$\delta^\alpha \in W_0^{1,q'}(\Omega) \quad \text{and} \quad \int_\Omega \left|\nabla\delta^\alpha\right|^{q'} dx \leqslant \int_\Omega \delta^{-\beta q'}(x)\,dx < +\infty.$$

Choosing as a test function $\varphi = \delta^\alpha$ in relation (32)

$$0 \leqslant \int_\Omega f\delta^\alpha\,dx \leqslant c\left[\int_\Omega |\nabla v|^q\,dx + \int_\Omega |v|^q\,dx + \int_\Omega |v|\,dx\right] \tag{34}$$

which shows the result. □

Next, we want to analyze some specific case, namely when we “symmetrize” the equation.

Unfortunately, the usual trick consisting to compare $\underset{\sim}{v}$ (when $f \geqslant 0$) with a radial decreasing function U associated to $\underset{\sim}{f}$ radial decreasing rearrangement of f, does not give any information for the integrability of v either its gradient (see Lemma 6). The following remark explains partly this fact.

Remark 2. In general, when we consider the ball $\widetilde{\Omega}$ having the same measure $|\Omega|$ than Ω, then the distance to the boundary $\delta(x) = \delta_{\widetilde{\Omega}}(x)$ is given by

$$\begin{cases} \delta_{\widetilde{\Omega}} = \alpha_N^{-\frac{1}{N}}|\Omega|^{\frac{1}{N}} - |x|, \\ x \in \widetilde{\Omega}. \end{cases}$$

Setting $R = \alpha_N^{-\frac{1}{N}}|\Omega|^{\frac{1}{N}}$, if $\underset{\sim}{f} \in L^1(\widetilde{\Omega}, \delta_{\widetilde{\Omega}})$, $f \geqslant 0$ then $f \in L^1(\Omega)$. Indeed

$$\begin{aligned} \int_\Omega f\,dy = \int_{\widetilde{\Omega}} \underset{\sim}{f}(x)\,dx &\leqslant \int_{\{|x| \leqslant \frac{R}{2}\}} \underset{\sim}{f}(x)\,dx + \int_{\{\frac{R}{2} < |x| \leqslant R\}} \underset{\sim}{f}(x)\,dx \\ &\leqslant \frac{2}{R}\int_{\widetilde{\Omega}} \underset{\sim}{f}(x)\delta_{\widetilde{\Omega}}(x)\,dx + f_*\left(\alpha_N\left(\frac{R}{2}\right)^N\right)|\Omega| \\ &< +\infty. \end{aligned}$$

The situation is different with respect to the increasing symmetric rearrangement $\widetilde{f}$ of f defined through the scalar increasing rearrangement of f.

We shall use

Lemma 4. *Assume that* $\mathrm{Max}_{\overline{\Omega}}\,\delta(x) = 1$ *and* $|\Omega| = \alpha_N$. *Then*

(i) $\delta_*(s) = 1 - \frac{s^{\frac{1}{N}}}{\alpha_N^{\frac{1}{N}}}$, $s \in [0, |\Omega|]$, *and up to a translation* Ω *is equal to the unit ball.*

(ii) $\forall f \in L^1(\Omega,\delta)$, $f \geqslant 0$, *one has*

$$\begin{cases} f_* \in L^1(\Omega_*,\sigma), \\ \int_{\Omega_*} f_*(\sigma)\sigma\, d\sigma \leqslant N\alpha_N \int_\Omega f(x)\delta(x)\, dx. \end{cases}$$

Proof. Following the proof given in [14], we have

$$\delta_*(0) - \frac{s^{\frac{1}{N}}}{\alpha_N^{\frac{1}{N}}} \leqslant \delta_*(s) \leqslant \frac{|\Omega|^{\frac{1}{N}} - s^{\frac{1}{N}}}{\alpha_N^{\frac{1}{N}}}. \tag{35}$$

Therefore, under the assumptions of the lemma, we have

$$\delta_*(0) = 1 = \frac{|\Omega|^{\frac{1}{N}}}{\alpha_N^{\frac{1}{N}}},$$

which shows (i).

While for (ii) we apply the Hardy–Littlewood inequality to derive

$$\int_\Omega \widetilde{f}(x)\,\underset{\sim}{\delta}(x)\, dx \leqslant \int_\Omega f(x)\delta(x)\, dx. \tag{36}$$

But, one has

$$\begin{aligned}
\int_\Omega \widetilde{f}(x)\,\underset{\sim}{\delta}(x)\, dx &= \int_\Omega f_*\big(|\Omega| - \alpha_N|x|^N\big)\cdot \delta_*\big(\alpha_N|x|^N\big)\, dx \\
&= N\alpha_N \int_0^1 f_*\big(\alpha_N\big(1-r^N\big)\big)\delta_*\big(\alpha_N r^N\big) r^{N-1}\, dr \quad \big(\text{setting } t = \alpha_N r^N\big) \\
&= \int_0^{\alpha_N} f_*(\alpha_N - t)\delta_*(t)\, dt \\
&= \frac{1}{\alpha_N^{\frac{1}{N}}} \int_0^{\alpha_N} f_*(\alpha_N - t)\cdot \big(\alpha_N^{\frac{1}{N}} - t^{\frac{1}{N}}\big)\, dt \\
&= \frac{1}{\alpha_N^{\frac{1}{N}}} \int_0^{\alpha_N} f_*(\alpha_N - t)\cdot \frac{(\alpha_N - t)}{P_N(t)}\, dt,
\end{aligned} \tag{37}$$

where, for $t \in [0, \alpha_N]$,

$$P_N(t) = \frac{\alpha_N - t}{\alpha_N^{\frac{1}{N}} - t^{\frac{1}{N}}}$$

is a polynomial function increasing in t and

$$\alpha_N^{1-\frac{1}{N}} = P_N(0) \leqslant P_N(t) \leqslant P_N(\alpha_N) = N\alpha_N^{1-\frac{1}{N}}.$$

Thus from (36), we have

$$\frac{1}{\alpha_N^{\frac{1}{N}}} \frac{1}{P_N(\alpha_N)} \int_0^{\alpha_N} f_*(\sigma)\sigma\, d\sigma \leqslant \int_\Omega \widetilde{f}(x)\, \underset{\sim}{\delta}(x)\, dx. \tag{38}$$

From (36) and (38) we get statement (ii). □

5. Some results on a ball: f is replaced by its increasing rearrangement $\widetilde{f}$

The aim of this section is to show that there are functions v whose data are in $L^1(\Omega, \delta)$ for which we have only the regularity $W^{1,1}(\Omega)$.

Setting

$$L^1(\Omega, \delta^{1^-}) = \bigcup_{0 \leqslant \alpha < 1} L^1(\Omega, \delta^\alpha),$$

a natural question concerns the global differentiability of v on the entire Ω when $f \in L^1(\Omega, \delta) \setminus L^1(\Omega, \delta^{1^-})$.

A partial answer can be given if Ω is a ball and when $L = -\Delta$. In this case, we have an estimate of the gradient of the Green function given in [10]. We have

Proposition 1. *Assume that $L = -\Delta$. If $f \in L^1(\Omega, \delta|\mathrm{Ln}\,\delta|)$ then the function v solution of $(DG_L(\Omega))$ is in $W_0^{1,1}(\Omega)$.*

Moreover, there exists a constant $c > 0$:

$$|\nabla v|_{L^1} \leqslant c|f|_{L^1(\Omega, \delta|\mathrm{Ln}\,\delta|)}.$$

Proof. The function $v_k \in W^{2,p}(\Omega \cap H_0^1(\Omega)$, $p \in [1, +\infty[$ solution of $-\Delta v_k = f_k$ satisfies that

$$v_k(x) = \int_\Omega G(x, y) f_k(y)\, dy,$$

where G is the Green function associated to the Dirichlet problem. According to [10], there exists a constant $c > 0$ such that

$$|\nabla G|(x, y) \leqslant c|x - y|^{1-N} \min\left\{1, \frac{\delta(y)}{|x - y|}\right\}. \tag{39}$$

By Fubini's theorem, one has

$$\int_\Omega |\nabla v_k|(x)\,dx \leqslant \int_\Omega \big|f_k(y)\big|\bigg[\int_\Omega |\nabla G|(x,y)\,dx\bigg]\,dy \tag{40}$$

and by the estimates (39), we have

$$\int_\Omega |\nabla G|(x,y)\,dx \leqslant c \int_{\{x:\ |x-y|\leqslant\delta(y)\}} |x-y|^{1-N}\,dx + c\delta(y) \int_{\{x:\ |x-y|>\delta(y)\}} |x-y|^{-N}\,dx. \tag{41}$$

Thus

$$\int_\Omega |\nabla G|(x,y)\,dx \leqslant c\delta(y) + c\delta(y)\big|\mathrm{Ln}\,\delta(y)\big|. \tag{42}$$

From relations (40) and (42), we deduce

$$\int_\Omega |\nabla v_k|(x)\,dx \leqslant c\int_\Omega \big|f(y)\big|\delta(y)\big|\mathrm{Ln}\,\delta(y)\big|\,dy. \tag{43}$$

This shows that $(v_k)_{k\geqslant 1}$ is a Cauchy sequence in $W_0^{1,1}(\Omega)$.

Thus $v \in W_0^{1,1}(\Omega)$. □

We recall the following result which can be obtained by some direct integrations (see for instance [11,7]).

Lemma 5. *Let $f \in L^1(\Omega,\delta)$, $f \geqslant 0$ and let for $n \in \mathbb{N}$, $T_n(f) = \min(f,n) \doteq f_n$. Then the sequence $(U_n)_{n\geqslant 0}$ defined on Ω by*

$$U_n(x) = \frac{1}{N^2\alpha_N^{\frac{2}{N}}} \int_{\alpha_N|x|^N}^{\alpha_N} \bigg[\sigma^{-2(1-\frac{1}{N})}\int_0^\sigma f_{n*}(t)\,dt\bigg]\,d\sigma,$$

is the unique solution of

$$\begin{cases} -\Delta U_n(x) = \underset{\sim}{f_n}(x) = f_{n*}\big(\alpha_N|x|^N\big), & x \in \Omega,\\ U_n = 0 \quad \text{on } \partial\Omega, \end{cases}$$

$U_n \in W_0^{1,q}(\Omega)$, $\forall q < +\infty$.

Another lemma that shall explain the difference between the results when $f \in L^1(\Omega)$ and $f \in L^1(\Omega,\delta)$ is the following necessary and sufficient condition.

Lemma 6. *Under the same assumptions as in Lemma* 5, *we have* $f \in L^1(\Omega)$ *if and only if* $\lim_{n\to+\infty} U_n = U$ *is in* $L^1(\Omega)$.

And in this case (*i.e.* $f \in L^1(\Omega)$), *the function* U *is the unique solution of*

$$\begin{cases} -\Delta U = \underset{\sim}{f} & \text{in } \Omega, \\ U \in W_0^{1,q}(\Omega), & 1 \leqslant q < \dfrac{N}{N-1}. \end{cases}$$

Proof. We first note that $f_{n*} = T_n(f_*) = \min(f_*, n)$, thus by Beppo–Levi monotone convergence $\lim_{n\to+\infty} U_n(x) = U(x)$ exists in $[0, +\infty]$ for a.e. x, since

$$U_n(x) \leqslant U_{n+1}(x) \quad \forall x \in \overline{\Omega}.$$

The first part is well known since if $f \in L^1(\Omega)$ then $\underset{\sim}{f} \in L^1(\Omega)$ and therefore, the unique solution of the Dirichlet problem is U.

Conversely, assume that $0 \leqslant \int_\Omega U(x)\,dx < +\infty$, one has for all $n \geqslant 1$, using integration by parts

$$\begin{aligned} \int_\Omega U(x)\,dx &\geqslant \int_\Omega U_n(x)\,dx \\ &= \int_0^{\alpha_N} s^{-1+\frac{1}{N}} \left(\int_0^s f_{n*}(t)\,dt \right) ds \\ &= N \int_0^{\alpha_N} \left(\alpha_N^{\frac{1}{N}} - s^{\frac{1}{N}}\right) f_{n*}(s)\,ds. \end{aligned} \tag{44}$$

From relation (44), one has for $0 < \varepsilon^{\frac{1}{N}} \leqslant \frac{\alpha_N^{\frac{1}{N}}}{2}$

$$\int_0^\varepsilon f_{n*}(s)\,ds \leqslant \frac{2}{N\alpha_N^{\frac{1}{N}}} |U|_{L^1(\Omega)} \tag{45}$$

and

$$\int_\varepsilon^{\alpha_N} f_{n*}(s)\,ds \leqslant f_{n*}(\varepsilon) \cdot (\alpha_N - \varepsilon) \leqslant f_*(\varepsilon)(\alpha_N - \varepsilon). \tag{46}$$

We note that since $f \in L_+^1(\Omega, \delta)$ then for all $s \in]0, |\Omega|]$, $0 \leqslant f_*(s) < +\infty$, in particular $f_*(\varepsilon) < +\infty$. Thus, for all $n \geqslant 0$

$$\int_\Omega f_n(x)\,dx \leqslant f_*(\varepsilon)(\alpha_N - \varepsilon) + \frac{2}{N\alpha_N^{\frac{1}{N}}} |U|_{L^1(\Omega)}.$$

This implies, by Fatou's lemma that

$$\int_\Omega f(x)\,dx \leqslant f_*(\varepsilon)(\alpha_N-\varepsilon)+\frac{2}{N\alpha_N^{\frac{1}{N}}}|U|_{L^1(\Omega)}. \qquad \square \tag{47}$$

Next, we want to prove the following theorem.

Theorem 5. *Let $h\in L^1(\Omega,\delta)$, $h\geqslant 0$. Then, the unique solution $\omega\in L^1(\Omega)$ of*

$$\begin{cases} -\Delta\omega(x)=\widetilde{h}(x)=h_*\big(\alpha_N-\alpha_N|x|^N\big), & \text{in } \Omega,\\ \omega(x)=0, \quad x\in\partial\Omega, \end{cases}$$

in the very weak sense given above belongs to $W_0^{1,1}(\Omega)$.

Moreover we have

$$|\omega|_{L^1(\Omega)}\leqslant \frac{1}{\alpha_N^{1+\frac{1}{N}}N}\big|h_*(\sigma)\sigma\big|_{L^1(\Omega_*)}\leqslant \frac{1}{\alpha_N^{\frac{1}{N}}}|h|_{L^1(\Omega,\delta)},$$

$$|\nabla\omega|_{L^1(\Omega)}\leqslant \frac{1}{N\alpha_N^{\frac{1}{N}}}\big|h_*(\sigma)\sigma\big|_{L^1(\Omega_*)}\leqslant |h|_{L^1(\Omega,\delta)}.$$

For this, we shall prove the following more general theorem which shows merely that for radial solution ω, one has the $W^{1,1}$-regularity.

Theorem 6. *Let f_0 be a given nonnegative measurable function on the interval Ω_* with $\sigma f_0(\sigma)\in L^1(\Omega_*)$.*

Then, $f\in L^1(\Omega,\delta_\Omega)$, with $f(x)=f_0(\alpha_N-\alpha_N|x|^N)$, and the unique generalized function $\omega\in L^1(\Omega)$ of $-\Delta\omega=f_0(\alpha_N-\alpha_N|x|^N)$ belongs to $W_0^{1,1}(\Omega)$.

Moreover we have

$$|\omega|_{L^1(\Omega)}\leqslant \frac{1}{\alpha_N^{1+\frac{1}{N}}N}\big|f_0(\sigma)\sigma\big|_{L^1(\Omega_*)}\leqslant \frac{1}{\alpha_N^{\frac{1}{N}}}|f|_{L^1(\Omega,\delta)},$$

$$|\nabla\omega|_{L^1(\Omega)}\leqslant \frac{1}{N\alpha_N^{\frac{1}{N}}}\big|f_0(\sigma)\sigma\big|_{L^1(\Omega_*)}\leqslant |f|_{L^1(\Omega,\delta)}.$$

Proof. Consider for $f\geqslant 0$, with $f(x)=f_0(\alpha_N-\alpha_N|x|^N)$. We first remark, arguing as in Lemma 4, that $\int_\Omega(1-|x|)f(x)\,dx$ is equivalent to $\int_0^{\alpha_N}\sigma f_0(\sigma)\,d\sigma$ and we have precisely

$$\alpha_N\int_\Omega f(x)\delta(x)\,dx\leqslant \int_0^{\alpha_N}\sigma f_0(\sigma)\,d\sigma\leqslant N\alpha_N\int_\Omega f(x)\delta(x)\,dx.$$

Thus, under the condition on f_0, one deduces that $f\in L^1(\Omega,\delta)$.

The function

$$\omega(x) = \frac{1}{N^2 \alpha_N^{\frac{2}{N}}} \int_{\alpha_N |x|^N}^{\alpha_N} \sigma^{-2(1-\frac{1}{N})} \left(\int_0^\sigma f_0(\alpha_N - t)\, dt \right) d\sigma$$

is $L^1(\Omega)$. Indeed, considering $f_{0n} = T_n(f_0)$ and the function

$$\overline{\omega}_n(x) = \int_{\alpha_N |x|^N}^{\alpha_N} \sigma^{-2(1-\frac{1}{N})} \left(\int_0^s f_{0n}(\alpha_N - t)\, dt \right) ds, \quad x \in \Omega,$$

one has by change of variables

$$\begin{aligned}
\int_\Omega \left|\overline{\omega}_n(x)\right| dx &= \int_0^{\alpha_N} \left[\int_s^{\alpha_N} \sigma^{-2(1-\frac{1}{N})} \left(\int_0^\sigma f_{0n}(\alpha_N - t)\, dt \right) \right] d\sigma \\
&= N \int_0^{\alpha_N} \left(\alpha_N^{\frac{1}{N}} - s^{\frac{1}{N}}\right) f_{0n}(\alpha_N - s)\, ds \\
&= N \int_0^{\alpha_N} \frac{(\alpha_N - s) f_{0n}(\alpha_N - s)}{P_N(s)}\, ds \qquad (48)
\end{aligned}$$

with $P_N(s) = \frac{\alpha_N - s}{\alpha_N^{\frac{1}{N}} - s^{\frac{1}{N}}}$, $s \in [0, \alpha_N[$. Thus from (48), we deduce

$$\int_\Omega \left|\overline{\omega}_n(x)\right| dx \leqslant \frac{N}{P_N(0)} \int_0^{\alpha_N} f_0(\sigma)\sigma\, d\sigma. \qquad (49)$$

Letting $n \to +\infty$, in relation (49), we deduce from Fatou's lemma

$$\int_\Omega \left|\omega(x)\right| dx \leqslant \frac{1}{N \alpha_N^{1+\frac{1}{N}}} \int_0^{\alpha_N} f_0(\sigma)\sigma\, d\sigma.$$

The same analysis shows that for $(j, n) \in \mathbb{N}^2$, one has

$$\int_\Omega \left|\nabla(\overline{\omega}_n - \overline{\omega}_j)(x)\right| dx \leqslant \alpha_N^{\frac{1}{N}} \int_0^{\alpha_N} s^{-1+\frac{1}{N}} \left| \int_0^s (f_{0n} - f_{0j})(\alpha_N - t)\, dt \right| ds. \qquad (50)$$

From the latter, we derive

$$\int_{\Omega} \left|\nabla(\overline{\omega}_n - \overline{\omega}_j)(x)\right| dx \leqslant \frac{N\alpha_N^{\frac{1}{N}}}{P_N(0)} \int_0^{\alpha_N} (\alpha_N - s)|f_{0n} - f_{0j}|(\alpha_N - s)\, ds$$
$$\leqslant N\alpha_N^{\frac{2}{N}-1} \int_0^{\alpha_N} \sigma\left|(f_{0n} - f_{0j})\right|(\sigma)\, d\sigma.$$

By Lebesgue dominate theorem, we deduce that

$$\lim_{\substack{n\to+\infty\\ j\to+\infty}} \int_0^{\alpha_N} \sigma\left|(f_{0n} - f_{0j})\right|(\sigma)\, d\sigma = 0.$$

Thus $\overline{\omega}_n$ is a Cauchy sequence in $W_0^{1,1}(\Omega)$. Therefore $\overline{\omega}$ is $W_0^{1,1}(\Omega)$ and so is ω.

Moreover, we have the identity

$$\int_{\Omega} \left|\nabla\overline{\omega}(x)\right| dx = N\alpha_N{}^{\frac{1}{N}} \int_0^{\alpha_N} \frac{(\alpha_N - t) f_0(\alpha_N - t)}{P_N(t)}\, dt. \tag{51}$$

From the latter, we derive

$$|\nabla\omega|_{L^1(\Omega)} \leqslant c_{3N}|f|_{L^1(\Omega,\delta)} \quad \text{with } c_{3N} = \frac{1}{N\alpha_N}. \tag{52}$$

Since one has

$$-\Delta\omega_n = f_{0n}\big(\alpha_N - \alpha_N|x|^N\big), \quad \omega_n \in H_0^1(\Omega),$$

this implies that ω is a solution of $DG(\Omega)$. □

As a complement for Theorem 4, we can make precise the necessary and sufficient condition for radial solution as in the above theorem. This will allow us to construct easily some examples for the applications.

Lemma 7. *Let $q \in [1, N'[$. Then the function ω given in Theorem* 6 *is in $W_0^{1,q}(\Omega)$ if and only if we have*

$$\int_0^{\alpha_N} \sigma f_0(\sigma)\left(\int_\sigma^{\alpha_N} f_0(t)\, dt\right)^{q-1} d\sigma = \int_0^{\alpha_N}\left(\int_\sigma^{\alpha_N} f_0(t)\, dt\right)^q d\sigma \quad \textit{is finite}.$$

Proof. Assume first that $f_0 \in L^\infty(\Omega_*)$. One has for any $q \in [1, N'[$

$$\int_{\Omega} \left|\nabla\omega(x)\right|^q dx = \gamma_N \int_0^{\alpha_N} s^{-\frac{q}{N'}} \left(\int_0^s f_0(\alpha_N - t)\, dt\right)^q dx$$

$$= \gamma'_N \int_0^{\alpha_N} \left(\alpha_N^{1-\frac{q}{N'}} - s^{1-\frac{q}{N'}}\right) f_0(\alpha_N - s)\left(\int_0^{s} f_0(\alpha_N - t)\,dt\right)^{q-1} ds$$

$$= \gamma'_N \int_0^{\alpha_N} \left[\frac{\alpha_N^{1-\frac{q}{N'}} - (\alpha_N - \sigma)^{1-\frac{q}{N'}}}{\sigma}\right]\sigma f_0(\sigma)\left(\int_\sigma^{\alpha_N} f_0(t)\,dt\right)^{q-1} d\sigma.$$

One has two constants $c_1 > c_0 > 0$ such that $\forall\sigma \in [0, \alpha_N]$

$$c_0 \leqslant \frac{\alpha_N^{1-\frac{q}{N'}} - (\alpha_N - \sigma)^{1-\frac{q}{N'}}}{\sigma} \leqslant c_1.$$

Indeed

$$\alpha_N^{1-\frac{q}{N'}} - (\alpha_N - \sigma)^{1-\frac{q}{N'}} = \alpha_N^{1-\frac{q}{N'}}\left[1 - \left(1 - \frac{\sigma}{\alpha\ N}\right)^{1-\frac{q}{N'}}\right].$$

The last function is equivalent to

$$\alpha_N^{-\frac{q}{N'}}\left(1 - \frac{q}{N'}\right)\sigma \quad \text{as } \sigma \to 0.$$

Therefore the quotient

$$\frac{\alpha_N^{1-\frac{q}{N'}} - (\alpha_N - \sigma)^{1-\frac{q}{N'}}}{\sigma} \underset{\sigma\to 0}{\approx} \alpha_N^{-\frac{q}{N'}}\left(1 - \frac{q}{N'}\right).$$

Thus, one has the equivalence

$$\int_\Omega \left|\nabla\omega(x)\right|^q dx \approx \int_0^{\alpha_N} \sigma f_0(\sigma)\left(\int_\sigma^{\alpha_N} f_0(t)\,dt\right)^{q-1} d\sigma.$$

Since $f_0(t), t \in L^1(0, \alpha_N)$, then by integration by parts, the last integral is equal to

$$\int_0^{\alpha_N}\left(\int_\sigma^{\alpha_N} f_0(t)\,dt\right)^q d\sigma = I(f).$$

We have shown that there are two constants k_{1N} and k_{2N} depending only on N and q such that

$$k_{1N} I(f) \leqslant \int_\Omega |\nabla\omega|^q\,dx \leqslant k_{2N} I(f). \tag{53}$$

If $f \in L^1(\Omega, \delta)$, $f \geqslant 0$, $f_n = T_n(f) \in L^1(\Omega, \delta)$ and with the expression of ω_n and ω, we have by Beppo–Levi monotone convergence

$$\lim_{n \to +\infty} \int_\Omega |\nabla \omega_n|^q \, dx = \int_\Omega |\nabla \omega|^q \, dx \tag{54}$$

and

$$\lim_{n \to +\infty} \int_0^{\alpha_N} \left(\int_\sigma^{\alpha_N} f_{0n}(t)\, dt \right)^q d\sigma = \int_0^{\alpha_N} \left(\int_\sigma^{\alpha_N} f_0(t)\, dt \right)^q d\sigma. \tag{55}$$

These numbers might be infinite. Thus (53) is true for $f \in L^1(\Omega, \delta)$.

This gives the equivalence. □

We shall end this section by a few examples of applications of the above results.

Corollary 7.1. *Let Ω be the unit ball of $\mathbb{R}^N$ and $q \in [1, \frac{N}{N-1}[$ for $\gamma \in [1, 2[$, we consider*

$$f(x) = \frac{1}{(1 - |x|^N)^\gamma}.$$

Then

$$f \in L^1(\Omega, \delta) \quad \textit{and} \quad f \notin L^1(\Omega).$$

Moreover

- *if $\gamma \in [1 + \frac{1}{q}, 2[$ then the function ω given in Theorem* 5 is not *in $W_0^{1,q}(\Omega)$;*
- *if $\gamma \in [1, 1 + \frac{1}{q}[$ then the function $\omega \in W_0^{1,q}$ with $q \in [1, \min(\frac{1}{\gamma-1}, \frac{N}{N-1})[$.*

Proof. One has

$$f_*(\sigma) = \frac{\beta_N}{\sigma^\gamma}, \quad \sigma \in [0, \alpha_N].$$

The necessary and sufficient condition can be written as

$$I(\gamma) = \int_0^{\alpha_N} \left(\sigma^{1-\gamma} - \alpha_N^{1-\gamma}\right)^q d\sigma$$

is finite if and only if $\omega \in W_0^{1,q}(\Omega)$. And

$$I(\gamma) \text{ is finite} \quad \text{if and only if} \quad \int_0^{\alpha_N} \sigma^{(1-\gamma)q}\, d\sigma \text{ is finite,}$$

and

$$\int_0^{\alpha_N} \sigma^{(1-\gamma)q}\,d\sigma \text{ is finite} \quad \text{if and only if} \quad \gamma < 1 + \frac{1}{q}. \qquad \square$$

Corollary 7.2. *Let* $g(\sigma) = \frac{4\alpha_N}{\sigma|\mathrm{Ln}\frac{\sigma}{4\alpha_N}|^\gamma}$, *with* $\gamma > 1$ *we set* $f(\sigma) = -g'(\sigma)$, $\sigma \in]0, \alpha_N[$. *Then*

(1) $g \in L^1(]0, \alpha_N[)$ *and* $g \notin L^q(]0, \alpha_N[)$ *for all* $q > 1$.
(2) *Setting* $h(x) = f(\alpha_n - \alpha_N|x|^N)$, $x \in \Omega$, $h \in L^1(\Omega, \delta|\mathrm{Ln}\,\delta|)$, *but h is not in* $L^1(\Omega, \delta^\alpha)$, *for any* $\alpha \in [0, 1[$.
(3) *The generalized function* $\omega \in L^1(\Omega)$ *solution of* $-\Delta\omega = h$ *belongs to* $W_0^{1,1}(\Omega)$ *but not to* $W_0^{1,q}(\Omega)$ *for* $q > 1$.

Proof. (1) $\int_0^{\alpha_N} g(\sigma)^q\,d\sigma \approx \int_{\mathrm{Ln}\,4}^{+\infty} \exp((q-1)\sigma)\sigma^{-\gamma}\,d\sigma$ from which we have the result.

(2) We set $X(\sigma) = |\mathrm{Ln}\,\frac{\sigma}{4\alpha_N}| = \mathrm{Ln}\,\frac{4\alpha_N}{\sigma}$ for $\sigma \in]0, \alpha_n]$, $Y(\sigma) = 1 - \frac{\gamma}{X(\sigma)}$.

By a straightforward computation, we have

$$g'(\sigma) = -\frac{g(\sigma)}{\sigma} Y(\sigma).$$

Thus $f(\sigma) = \frac{g(\sigma)}{\sigma} Y(\sigma)$ and for $\alpha \in [0, 1]$ we have:

$$J_\alpha = \int_\Omega |h|(x)\big(1 - |x|\big)^\alpha\,dx = \int_0^{\alpha_N} \frac{\sigma^\alpha |f|(\sigma)\,d\sigma}{P_N^\alpha(\alpha_N^{-\frac{1}{N}}(\alpha_N - \sigma)^{\frac{1}{N}})},$$

with $P_N^\alpha(t) = (\frac{1-t^N}{1-t}\alpha_N)^\alpha$, $t \in [0, 1]$.

Since $\inf_{t\in[0,1]} P_N^\alpha(t) > 0$, we deduce that

$$J_\alpha \approx \int_0^{\alpha_N} \sigma^\alpha \big|f(\sigma)\big|\,d\sigma \approx \int_0^{\alpha_N} \big|Y(\sigma)\big| g(\sigma)\sigma^{\alpha-1}\,d\sigma.$$

Let us introduce $\sigma_N = 4\alpha_N \exp(-\gamma)$, then

$$0 < Y(\sigma) \leqslant 1 \quad \text{if } \sigma \in [0, \sigma_N[\quad \text{and} \quad Y(\sigma) < 0 \quad \text{for } \sigma > \sigma_N, \quad Y(\sigma_N) = 0.$$

If $\alpha = 1$ since $\mathrm{Max}_{\sigma\in[0,\alpha_N]} |Y(\sigma)|$ is finite then

$$J_1 \leqslant c \int_0^{\alpha_N} g(\sigma)\,d\sigma < +\infty.$$

If $0 \leqslant \alpha < 1$, then we have

$$J_\alpha \geqslant c \int_0^{\varepsilon\alpha_N} \left|Y(\sigma)\right| g(\sigma)\sigma^{\alpha-1}\, d\sigma,$$

with $0 < \varepsilon < 1$, $0 < \varepsilon\alpha_N < \frac{1}{2}\inf(\alpha_N, \sigma_N)$. From the latter, we have

$$J_\alpha \geqslant c \int_{\mathrm{Ln}\,\frac{4}{\varepsilon}}^{+\infty} \exp\bigl((1-\alpha)\sigma\bigr)\sigma^{-\gamma}\, d\sigma = +\infty.$$

To show that $h \in L^1(\Omega, \delta|\mathrm{Ln}\,\delta|)$, we start with the case $\gamma \leqslant \mathrm{Ln}\,4$ then $f'(\sigma) \leqslant 0$ for all $\sigma \in]0, \alpha_N[$, then $f(\sigma) = f_*(\sigma)$.

Since

$$\begin{aligned}\int_\Omega |h|(x)\delta(x)\left|\mathrm{Ln}\,\delta(x)\right| dx &\leqslant c \int_0^{\alpha_N} f(\sigma)\sigma\, d\sigma \\ &= c\int_0^{\alpha_N} f_*(\sigma)\sigma\, d\sigma \leqslant c \int_\Omega f(x)\delta(x)\, dx < +\infty,\end{aligned}$$

if $\gamma > 2$, we have (arguing as before):

$$\begin{aligned}&\int_\Omega \left|h(x)\right|\delta(x)\left|\mathrm{Ln}\,\delta(x)\right| dx \\ &\quad\leqslant c\int_\Omega \left|h(x)\right|\delta(x)\, dx + c\int_0^{\alpha_N} g(\sigma)\, d(\sigma) + c\int_{\mathrm{Ln}\,4}^{+\infty} \sigma^{(1-\gamma)}\, d\sigma < +\infty,\end{aligned}$$

thus $h \in L^1(\Omega, \delta|\mathrm{Ln}\,\delta|)$.

(3) According to Theorem 6 or Proposition 1, we then have that the very weak solution $-\Delta\omega = h$, $\omega \in L^1(\Omega)$ is in $W_0^{1,1}(\Omega)$. Since $h \notin L^1(\Omega, \delta^\alpha)$ then ω does not belong to $W_0^{1,q}(\Omega)$ for all $q > 1$. □

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References

[1] C. Bennett, R. Sharpley, Interpolation of Operators, Academic Press, London, 1983.

[2] I. Birindelli, F. Demengel, Comparison principle and Liouville type results for singular fully nonlinear operators, Ann. Fac. Sci. Toulouse Math. (6) 13 (2) (2004) 261–287.

[3] L. Boccardo, T. Gallouët, Non-linear elliptic and parabolic equations involving measure as data, J. Funct. Anal. 87 (1989) 149–169.

[4] H. Brezis, Personal communication to J.I. Díaz: Une équation semi-linéaire avec conditions aux limites dans L^1, unpublished.

[5] H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa, Blow up for $u_t - \Delta u = g(u)$ revisited, Adv. Differential Equations 1 (1996) 73–90.

[6] X. Cabré, Y. Martel, Weak eigenfunctions for the linearization of extremal elliptic problems, J. Funct. Anal. 156 (1998) 30–56.

[7] J.I. Díaz, Nonlinear Partial Differential Equations and Free Boundaries, Res. Notes Math., vol. 106, Pitman, London, 1985.

[8] J.I. Díaz, J.M. Rakotoson, On very weak solutions of semilinear elliptic equations with right hand side data integrable with respect to the distance to the boundary, in preparation.

[9] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, 1983.

[10] H.-Ch. Grunau, G. Sweers, Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions, Math. Nachr. 307 (1997) 89–102.

[11] L. Korkut, M. Pasic, D. Zubrinić, A singular ODE related to quasilinear elliptic equations, Electron. J. Differential Equations 12 (2000) 1–37.

[12] P. Quittner, P. Souplet, Superlinear Parabolic Problems, Birkhäuser, Basel, 2007.

[13] J.M. Rakotoson, Quasilinear elliptic problems with measure as data, Differential Integral Equations 4 (3) (1991) 449–457.

[14] J.M. Rakotoson, Réarrangement Relatif: un instrument d'estimation dans les problèmes aux limites, Springer-Verlag, Berlin, 2008.

[15] J.E. Rakotoson, J.M. Rakotoson, Analyse fonctionnelle appliquée aux équations aux dérivées partielles, P.U.F., Paris, 1999.

[16] C. Simader, On Dirichlet's Boundary Value Problem and L^p Theory Based on Generalization of Gårding's Inequality, Lecture Notes in Math., vol. 268, Springer-Verlag, New York, 1972.

[17] L. Veron, Singularities of Solutions of Second Order Quasilinear Equations, Longman, Edinburgh Gate, Harlow, 1995, pp. 176–180.